ECN 710 : Advanced Macroeconomics

Chapter 2: Solow Growth Model

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Plan

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- Canada at the beginning of the 20th century!
- Evolution of Countries Over Time: A Historical Perspective

The Solow Model

- The previous video raises two fundamental questions:
  1. How can a country initiate a growth process that will lead it to a higher level of GDP per capita?
  2. Why are some countries rich and others poor?
- This chapter presents a model, the **Solow Model** (1956, QJE), that provides some initial answers.
- We will show how the long-term evolution of income and consumption per worker in a country is affected by:
  - the savings rate and investment,
  - technological progress and the population growth rate.

Economic Environment

- The Solow model is a dynamic general equilibrium model.
- The economy is assumed to be closed and decentralized.
- The agents:
  - Households,
  - Firms.
- There is a final good, which is produced with two factors of production: capital and labor.
- Time is discrete, $t \in \{0, 1, 2, ...\}$.

The Household

- Representative household
  - Demands the final good.
  - Owns the factors of production (i.e. capital and labor).
  - Saves a constant fraction $\mathbf{s}$ of disposable income.

Firm: Technology and Production (1/4)

- All firms have access to the same production technology:

  $\Longrightarrow$ the economy admits a representative firm.

- The aggregate production function for the unique final good is:

$$Y_t = F(K_t, L_t, A_t) \tag{1}$$

- K_t and L_t represent the demand for capital and labor at time t.
- A_t is the technology at time t.
- Technology is free, publicly available as a non-exclusive and non-rival good.

Assumptions: We assume that $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is

- **continuous**, twice differentiable in K and L, and satisfies:

$$\begin{aligned} F_K(K,L,A) &\equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K,L,A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0 \\ F_{KK}(K,L,A) &\equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K,L,A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0 \end{aligned} \tag{2}$$

- with **constant returns to scale** in K and L, i.e., homogeneous of degree 1.
- The marginal products of capital and labor increase with the other factor, i.e., capital and labor are complementary:

$$F_{KL}(K,L,A) = F_{LK}(K,L,A) \equiv \frac{\partial^2 F}{\partial K \partial L}(K,L,A) > 0$$

- with $F_{KL}(K,L,A) = F_{LK}(K,L,A)$ by Schwarz's theorem.

Firm: Technology and Production (3/4)

Definition: Homogeneity

Let m be an integer. The function $F : \mathbb{R}^{p+2} \to \mathbb{R}$ is homogeneous of degree m in $(K, L) \in \mathbb{R}^2$ if and only if

$$F(\lambda K, \lambda L, A) = \lambda^m F(K, L, A), \qquad \forall \lambda \in \mathbb{R}_+ \text{ and } A \in \mathbb{R}^p. \tag{3}$$

Theorem 1 (Euler's Theorem).

Let $F : \mathbb{R}^{p+2} \to \mathbb{R}$ be a differentiable function with partial derivatives F_K and F_L, and **homogeneous** of degree m in K and L. Then:

① $mF(K, L, A) = F_K(K, L, A) \times K + F_L(K, L, A) \times L \quad \forall K, L \in \mathbb{R},\ A \in \mathbb{R}^p.$

② $F_K(K, L, A)$ and $F_L(K, L, A)$ are homogeneous of degree $m - 1$ in (K, L).

Proof of Euler's Theorem:

Since F is homogeneous of degree m in (K, L), for all $\lambda \in \mathbb{R}_+$, we have:

$$F(\lambda K, \lambda L, A) = \lambda^m F(K, L, A). \tag{4}$$

Since F is differentiable, differentiating (4) with respect to λ gives:

$$m\lambda^{m-1} F(K, L, A) = F_K(\lambda K, \lambda L, A)K + F_L(\lambda K, \lambda L, A)L$$

Substituting $\lambda = 1$, we obtain the first relation.

Differentiating (4) with respect to K and L and dividing both relations by $\lambda > 0$, we get:

$$F_K(\lambda K, \lambda L, A) = \lambda^{m-1} F_K(K, L, A)$$

$$F_L(\lambda K, \lambda L, A) = \lambda^{m-1} F_L(K, L, A)$$

Firm: Technology and Production (4/4)

The function $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ must satisfy the Inada conditions:

$$\begin{aligned} \lim_{K\to 0} F_K(K,L,A) = \infty \quad &\text{and} \quad \lim_{K\to\infty} F_K(K,L,A) = 0, \quad \forall L > 0 \\ \lim_{L\to 0} F_L(K,L,A) = \infty \quad &\text{and} \quad \lim_{L\to\infty} F_L(K,L,A) = 0, \quad \forall K > 0 \end{aligned} \tag{5}$$

$\Longrightarrow$ This is important for ensuring the existence of interior equilibria.

$\Longrightarrow$ All factors of production are necessary, i.e.

$$F(0,L,A) = F(K,0,A) = 0. \tag{6}$$

$\Longrightarrow$ Output is unbounded above, i.e.

$$\lim_{K\to\infty} F(K,L,A) = \infty \quad \text{and} \quad \lim_{L\to\infty} F(K,L,A) = \infty. \tag{7}$$

Production Function in Intensive Form

Let $y \equiv Y/L$ and $k \equiv K/L$ denote output and capital per worker, respectively.

- The production function F is homogeneous of degree 1, thus

$$y = F(K, L, A)/L = F(k, 1, A) \equiv f(k). \tag{8}$$

- By the definition of f and properties of F, we have the following:
  - $f(0) = 0$ and f is twice differentiable,
  - $\lim_{k \to 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 0$.
  - $F_K(K, L, A) = f'(k) \quad F_L(K, L, A) = f(k) - kf'(k)$.
  - f is increasing, $f'(k) > 0$ and concave, $f''(k) < 0$.

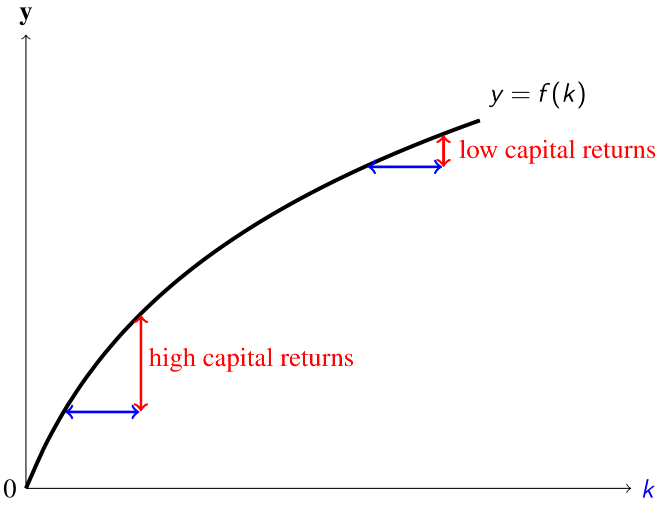


Figure: Production function: diminishing marginal returns

- Competitive Markets
  - Households and firms are price takers.
  - Prices equilibrate the markets.
- At each date t, households have $\bar{L}_t$ units of labor that they supply inelastically.
- The labor market equilibrium condition at time t can be expressed as follows:

$$L_t = \bar{L}_t$$

  where L_t is the labor demand by firms.
- Given the wage w_t, this equilibrium condition can also be written as

$$L_t \leq \bar{L}_t, \quad w_t > 0 \quad \text{and} \quad (L_t - \bar{L}_t) w_t = 0.$$

Market Structure, Endowments, and Equilibrium (2/3)

- Households also own capital and supply the quantity $\overline{K}_t > 0$, with $K_0 > 0$ being the initial capital stock.
- The equilibrium condition for the capital market is:

$$K_t = \overline{K}_t.$$

- Capital depreciates at rate δ, i.e., of 1 unit of capital used at time t, only $1 - \delta$ units remain for the next period.
- Let R_t be the real rent on capital at time t, and the interest rate received by households will be

$$r_t = R_t - \delta.$$

Market Structure, Endowments, and Equilibrium (3/3)

- The representative firm maximizes its profit by choosing the quantity of production factors

$$\max_{L_t \geq 0, K_t \geq 0} F(K_t, L_t, A_t) - w_t L_t - R_t K_t. \tag{9}$$

- The first-order conditions imply

$$R_t = F_K(K_t, L_t, A_t) = f'(k_t) \tag{10}$$

$$w_t = F_L(K_t, L_t, A_t) = f(k_t) - k_t f'(k_t) \tag{11}$$

- At the equilibrium of the Solow model, firms make zero profit, and in particular,

$$Y_t = w_t L_t + R_t K_t$$

This immediately follows from Euler's theorem for $m = 1$.

Capital Stock Movement Law

- The law of motion for the capital stock is

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{12}$$

  where I_t is the investment at time t.

- Since the economy is closed, the sum of global consumption and investment cannot exceed total production.

$$Y_t = C_t + I_t. \tag{13}$$

- Since the economy is closed (and there is no government spending),

$$S_t := Y_t - C_t = I_t. \tag{14}$$

Law of Motion of Capital Stock

- Households save a constant fraction s of their income,

$$S_t = sY_t. \tag{15}$$

$$C_t = Y_t - I_t = (1-s)Y_t. \tag{16}$$

- Thus, the **fundamental equation of the Solow model** is

$$\begin{aligned} K_{t+1} &= sY_t + (1-\delta)K_t \\ &= sF(K_t, L_t, A_t) + (1-\delta)K_t. \end{aligned} \tag{17}$$

- This is a nonlinear difference equation.

- The equilibrium of the Solow model is described by this equation, along with the laws of motion for L_t and A_t.

Law of Motion of Employment (Population)

- The demographic growth rate, denoted n, is exogenous and constant.
- Thus, at each date t, the number of households is given by

$$\overline{L}_t = (1+n)\overline{L}_{t-1} \quad \Longleftrightarrow \quad L_t = (1+n)^t \overline{L}_0$$

  with $\overline{L}_0$ being the initial population level.
- We normalize $\overline{L}_0 = 1$.

Equilibrium

Definition of Equilibrium

Given a sequence of $\{L_t, A_t\}_t^\infty$ and an initial capital stock K_0, an equilibrium path is a sequence of capital stocks, production levels, consumption levels, wages, and rental rates of capital $\{K_t, Y_t, C_t, w_t, R_t\}_t^\infty$ such that K_t satisfies the fundamental equation of the Solow model (17), Y_t is given by (1), C_t is given by (16), and w_t and R_t are given by (10) and (11), respectively.

- Equilibrium is defined as a complete trajectory of allocations and prices.
- Equilibrium does not refer to a static object, but rather specifies the trajectory of economic variables.

Steady State in the Solow Model

- Recall the fundamental dynamic equation of Solow

$$K_{t+1} = sF(K_t, L_t, A_t) + (1-\delta)K_t. \tag{18}$$

- Dividing this equation by L_t, we obtain

$$(n+1)k_{t+1} = sf(k_t) + (1-\delta)k_t. \tag{19}$$

⇒ Saving serves to replace depreciated capital, endow new workers with capital, and potentially increase the capital per worker.

- Equation (19) is also called the difference equation of the Solow model equilibrium.
- The other equilibrium quantities can be derived from k_t.

Steady State in the Solow Model

- Initially assume that $A_t = A$ is constant.

Definition

A steady-state equilibrium (without technological progress), also called a regular state, is an equilibrium path in which

$$k_t = k^*, \text{for all } t. \tag{20}$$

- The economy will tend toward this steady-state equilibrium over time (but it will never reach it in finite time).
- At the steady state, all aggregate variables (L_t, K_t, Y_t, C_t) and prices (w_t, R_t) grow at a constant rate, possibly zero.

Steady State in the Solow Model

The expressions for output and consumption per worker at the steady state are:

$$y^* = f(k^*) \quad \textbf{and} \quad c^* = (1-s)f(k^*)$$

Proposition

- k^* exists and is unique, satisfying
$$\frac{f(k^*)}{k^*} = \frac{\delta + n}{s} \tag{21}$$
- k^* and y^* increase with s and decrease with δ and n.
- c^* is non-monotonic with respect to s and decreases with δ and n.

Proof: : k^* is the steady-state capital if and only if it solves

$$(n+1)k^* = sf(k^*) + (1-\delta)k^* \tag{22}$$

This implies that

$$\phi(k^*) \equiv \frac{f(k^*)}{k^*} = \frac{\delta + n}{s}$$

The properties of f imply that ϕ is continuous and strictly decreasing, with

$$\phi'(k) = \frac{f'(k)k - f(k)}{k^2} = -\frac{F_L}{k^2} < 0,$$
$$\phi(0) = f'(0) = \infty \quad \text{and} \quad \phi(\infty) = f'(\infty) = 0,$$

Thus, equation (22) has a unique solution:

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right)$$

Recall that

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right).$$

By applying the formula for the derivative of the inverse function,

$$(\phi^{-1}(k))' = \frac{1}{\phi'[\phi^{-1}(k)]}$$

we find that k^* decreases with $(\delta + n)/s$ (since $\phi'(k) < 0$).

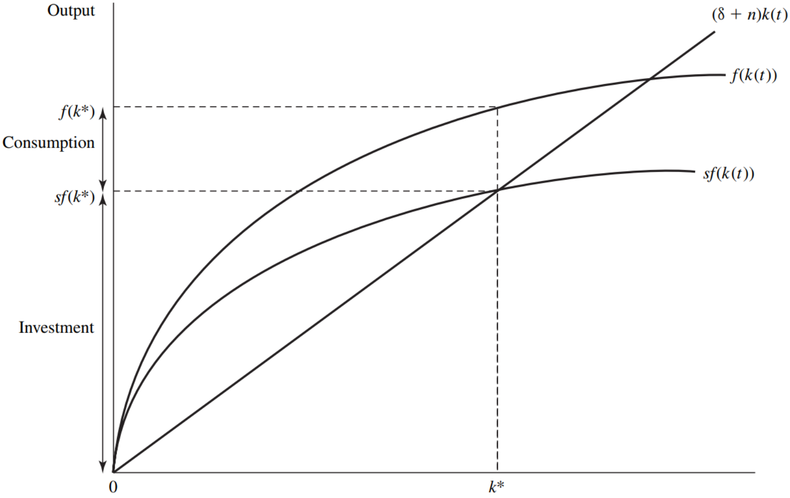
$y^* = f(k^*)$ and since f is increasing, y^* also decreases with $(\delta + n)/s$.

On the other hand, consumption is given by

$$c^* = (1 - s)f(k^*)$$

Thus, c^* decreases with $\delta + n$, but the effect of s is ambiguous.

Steady State in the Solow Model



Golden Rule

For each value of the savings rate s, there exists a unique value of k^*, and $\partial k^*(s)/\partial s > 0$. The steady-state consumption can be written as

$$c^*(s) = f(k^*(s)) - (n+\delta)k^*(s)$$

$c^*(s)$ is increasing for low values of s and decreasing for high values of s.

$$\frac{\partial c^*(s)}{\partial s} = \left[f'(k^*(s)) - (n+\delta)\right] \frac{\partial k^*(s)}{\partial s}$$

Let s_{gold} denote the savings rate that maximizes consumption, and k_{gold} the corresponding capital level. Then

$$f'(k_{gold}) = (n+\delta) \tag{23}$$

The equation (23) is called the golden rule of capital accumulation.

Golden Rule and Dynamic Inefficiency (1/4)

- In the Solow model, the savings rate is exogenous.
- An important question is what the appropriate savings rate should be.
- It is difficult to answer this until we have specified a detailed objective function (Lecture 3).
- One answer is to choose the savings rate that maximizes consumption, i.e., s_{gold}.
- A savings rate $s \geq s_{gold}$ is inefficient because higher consumption per capita could be achieved by reducing the savings rate.

- Suppose an economy with an saving rate $s_2 > s_{gold}$, then $k_2^* > k_{gold}$ and $c_2^* < c_{gold}$.
- If we reduce s_2 to s_{gold}, per capita consumption first increases and then decreases monotonically during the transition to its new steady-state value, c_{gold}.
- Since $c_2^* < c_{gold}$, consumption will be greater than c_2 at all moments during the transition and in the new steady-state.
- Therefore, when $s > s_{gold}$, the economy is in a situation of over-saving in the sense that per capita consumption at all points in time could be increased by reducing the saving rate.
- An economy that over-saves is said to be dynamically inefficient because its per capita consumption trajectory is lower than potential alternative trajectories at any point in time.

Golden Rule and Dynamic Inefficiency (3/4)

- If the economy has a saving rate $s_1 < s_{gold}$, then higher savings will increase per capita consumption in the steady state.
- However, this increase in the saving rate will reduce consumption today and for some time during the transition period.
- The result will therefore be seen as good or bad depending on how households weigh today's consumption against future consumption trajectories.
- We cannot judge whether an increase in the saving rate is appropriate in this situation until we have made specific assumptions about how agents discount the future.

Golden Rule and Dynamic Inefficiency (4/4)

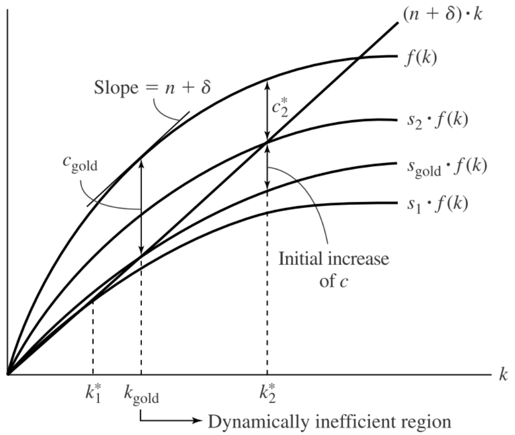


Figure: Golden Rule and Dynamic Inefficiency (Barro & Sala-i-Martin, p. 36)

Dynamic Transition (1/3)

The study of the transition dynamics answers the question:

Does an economy that is not at its steady state converge towards it?

In other words: **Will the economy eventually return to its steady state if an exogenous shock drives it away?**

Proposition

For any strictly positive initial level of capital per worker k_0, the economy converges towards its steady-state level.

- The growth rate is positive and decreases to zero if $k_0 < k^*$;
- The growth rate is negative and increases to zero if $k_0 > k^*$.

Proof. Let h be the function

$$h(k) = \frac{sf(k) + (1-\delta)k}{n+1}$$

h is strictly increasing and concave (and continuous and differentiable):

$$h'(k) = \frac{sf'(k) + (1-\delta)}{n+1} > 0 \quad \text{and} \quad h''(k) = \frac{sf''(k)}{n+1} < 0$$

We show that if $K_0 < k^*$ then K_t increases and stays to the left of k^*, i.e.

$$k_t < k^* \text{ and } \gamma_t \equiv \frac{\Delta k_t}{k_t} > 0, \forall t.$$

Since h is increasing, it is enough to apply h t times to the inequality $k_0 < k^*$ and obtain

$$k_t < k^*.$$

Moreover,

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = \frac{s}{n+1}\left[\frac{f(k_t)}{k_t} - \frac{n+\delta}{s}\right]$$

Recall that ϕ is decreasing.

$$k_t < k^* \implies \phi(k_t) - \frac{n+\delta}{s} > \phi(k^*) - \frac{n+\delta}{s} = 0$$

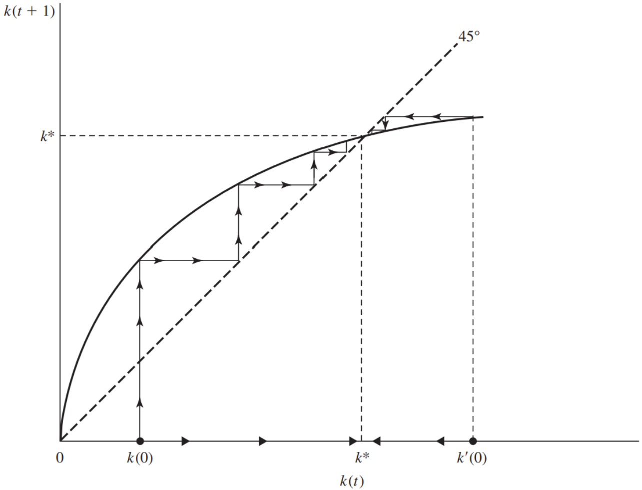
This shows that when $k_0 < k^*$, k_t is increasing.

Thus, when $k_0 < k^*$, K_t grows asymptotically towards k^*.

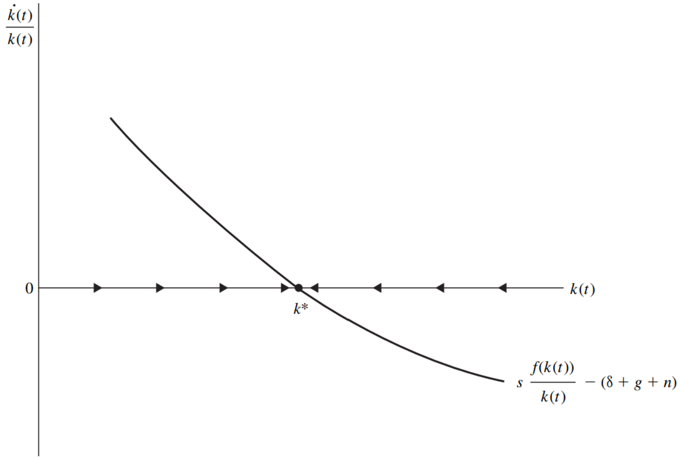
Since $k_{t+1} = h(k_t)$, and k_t converges, its limit is the unique fixed point of h, which is nothing but k^* by definition.

Symmetrically, it can be shown that if $k_0 > k^*$, then k_t decreases asymptotically towards k^*.

Dynamic Transition (2/3)



Dynamic Transition (3/3)



Solow Model in Continuous Time

- Consider the difference equation

$$x(t+1) - x(t) = g(x(t)). \tag{24}$$

where for each unit of time $t = 0, 1, 2, \ldots$, $g(x(t))$ describes the absolute growth of x between t and $t+1$.

- Equation (24) describes the variation of the variable x between two discrete points in time, t and $t+1$.
- Now consider the following approximation for $\Delta t \in [0,1]$

$$x(t+\Delta t) - x(t) \simeq \Delta t g(x(t)). \tag{25}$$

- If $\Delta t = 0$, equation (25) is just an identity, and if $\Delta t = 1$, we obtain equation (24).

Solow Model in Continuous Time

- This relation is a linear approximation between $\Delta t = 0$ and $\Delta t = 1$.
- This approximation will be relatively accurate if the distance between t and $t+1$ is not very large, so that $g(x) = g(x(t))$ for all $x \in [x(t), x(t+1)]$.
- By dividing both sides of (25) by Δt and taking the limit, we get:

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \dot{x}(t) \approx g(x(t)) \tag{26}$$

  where $\dot{x}(t) \equiv \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}$.
- Equation (26) is a differential equation representing the same dynamics as the difference equation (24).

Fundamental Equation of Solow: Continuous Time

- Nothing has changed on the production side,
  - equations (10) and (11) still give the factor prices,
  - now $w(t)$ and $R(t)$ are interpreted as the instantaneous wage rate and the rental rate of capital.
- Savings is again given by

$$S(t) = s\,Y(t).$$

- Consumption is given by

$$C(t) = (1 - s)Y(t). \tag{27}$$

- The labor factor (population) in continuous time,

$$L(t) = L(0)\exp(nt). \tag{28}$$

Equilibrium: Solow Model in Continuous Time (1/2)

- Recall that the capital stock per worker is

$$k(t) \equiv \frac{K(t)}{L(t)}$$

- Then

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} = \frac{\dot{K}(t)}{K(t)} - n. \tag{29}$$

- The fundamental equation of the Solow model, cf. Equation (18), in continuous time becomes:

$$\dot{K}(t) = sF(K(t), L(t), A) - \delta K(t) \tag{30}$$

- Using the properties of F, we get

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + n). \tag{31}$$

Equilibrium: Solow Model with Continuous Time (2/2)

Definition

Given the initial capital stock K_0, an equilibrium path is a sequence of capital stocks, labor, production levels, consumption levels, wages, and capital rental rates $\{K(t), L(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that $L(t)$ satisfies (28), $k(t) \equiv K(t)/L(t)$ satisfies (31), $Y(t)$ is given by the aggregate production function, $C(t)$ is given by (27), and $w(t)$ and $R(t)$ are given by (10) and (11).

- As before, the steady-state equilibrium implies that K_t remains constant at a certain level k^*.

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right) \quad \text{with} \quad \phi(k) = \frac{f(k)}{k}. \tag{32}$$

Dynamic Transition: Continuous Time (1/2)

Property

For an initial capital level $k(0) > 0$, the economy described by the Solow model in continuous time without technological change converges asymptotically to its steady state.

- $\dot{k}(t)/k(t) > 0$ and decreases toward zero if $k(0) < k^*$;
- $\dot{k}(t)/k(t) < 0$ and increases toward zero if $k(0) > k^*$.

The proof is identical to that provided for the discrete time model.

The growth rate of output is

$$\frac{\dot{y}}{y} = \frac{f'(k)}{f(k)}k = sf'(k) - (\delta + n)\alpha_k \tag{33}$$

where $\alpha_k(t) \equiv f'(K_t)K_t/f(K_t)$ is the share of capital income.

By differentiating equation (33) with respect to k, we find

$$\partial(\dot{y}/y)/\partial k = \left[\frac{f''(k)\cdot k}{f(k)}\right](\dot{k}/k) - \frac{(n+\delta)f'(k)}{f(k)}(1-\alpha_k) \tag{34}$$

- If $k(0) < k^*$, then $\dot{k}/k > 0$ and thus $\partial(\dot{y}/y)/\partial k \geq 0$.
- If $k(0) > k^*$, the sign of $\partial(\dot{y}/y)/\partial k$ is ambiguous, but it will be negative as we approach the steady state.

The Solow Model with Technological Progress (1/4)

- Production function (Uzawa's Theorems):

$$Y(t) = F(K(t), A(t)L(t)) \tag{35}$$

where technological progress makes labor more productive.

- Technological progress evolves at the rate $g > 0$, such that:

$$g = \frac{\dot{A}(t)}{A(t)}. \tag{36}$$

- The population continues to grow at the rate $n > 0$, with $n = \dot{L}(t)/L(t)$.
- Therefore, the fundamental equation of the Solow model becomes:

$$\dot{K}(t) = sF(K(t), A(t)L(t)) - \delta K(t) \tag{37}$$

- Let us now define $K(t)$ as capital per effective unit of labor, that is,

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \tag{38}$$

- Differentiating this expression with respect to time,

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K_t} - g - n. \tag{39}$$

- The output per effective unit of labor can be written as follows:

$$\begin{aligned} \hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} = F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\ &\equiv f(k(t)). \end{aligned}$$

The Solow Model with Technological Progress (3/4)

- The income per capita is then:

$$y(t) \equiv Y(t)/L(t) = A(t)\hat{y}(t) = A(t)f(k(t)) \tag{40}$$

- Thus, if $\hat{y}(t)$ is constant, income per capita, $y(t)$, will increase over time because $A(t)$ is increasing.
- We can no longer talk about a steady state where income per capita is constant.
- We now seek a balanced growth path, where income per capita increases at a constant rate.
- Some transformed variables, such as $\hat{y}(t)$ or $k(t)$ in (38), remain constant along the balanced growth path.

- The terms ”steady state”, ”regular state”, and ”balanced growth path” are used interchangeably.
- The equations (37) and (39) combined imply:

$$\frac{\dot{k}(t)}{k(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g + n). \tag{41}$$

- Which can also be written as:

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \tag{42}$$

- The only difference is the presence of g. Thus, k is no longer capital per labor, but capital per effective labor.

Proposition

A steady-state equilibrium (with technological progress and population growth) is a balanced path where

$$k(t) = k^{\star} \tag{43}$$

where $k^{\star}$ satisfies the following relation for all t:

$$\frac{f(k^{\star})}{k^{\star}} = \frac{\delta + n + g}{s}. \tag{44}$$

Production and consumption per worker grow at the rate g.

Proposition

For an initial level of capital per effective unit of labor $k(0) > 0$, the economy described by the Solow model with technological progress and population growth will asymptotically converge to its steady state:

$$k(t) \to k^{\star}.$$

Conclusion (1/3)

- This model provides a simple and workable framework to discuss capital accumulation and the implications of technological progress.
- It shows that without technological progress, there will be no sustained growth.
- It can generate growth in output per capita through technological progress, but only in an exogenous way.
- Technological progress is a ”black box.”
- Capital accumulation: determined by the savings rate, depreciation rate, and population growth rate. All these factors are exogenous.
- More work is needed to understand what lies within these ”black boxes.”

Conclusion (2/3)

Although the basic Solow model is old, it continues to be used in research.

- ✍ Brock, W. A., & Taylor, M. S. (2010). **The green Solow model**. *Journal of Economic Growth*, 15, 127-153.,

  Incorporates technological progress in emission reduction into the Solow model to estimate the environmental Kuznets curve.

- ✍ Durlauf, S. N., Kourtellos, A., & Minkin, A. (2001). **The local Solow growth model.** *European Economic Review*, 45(4-6), 928-940.

  Generalizes the empirical analysis of the Solow model by relaxing the assumption that all countries have the same production function.

- ✍ Ding, S., & Knight, J. (2009). **Can the augmented Solow model explain China's remarkable economic growth? A cross-country panel data analysis.** *Journal of Comparative Economics*, 37(3), 432-452.

  Uses panel data from 146 countries during 1980-2004 to examine how well the augmented Solow model explains China's rapid growth and the significant growth gap between China and other countries.

Conclusion (3/3)

- McDonald, S., & Roberts, J. (2002). **Growth and multiple forms of human capital in an augmented Solow model: A panel data investigation.** *Economics Letters*, 74(2), 271-276.

  They show that omitting health capital from augmented Solow growth models creates specification biases and that health capital significantly impacts economic growth rates.

- Bräuninger, M., & Pannenberg, M. (2002). **Unemployment and productivity growth: An empirical analysis within an augmented Solow model**. *Economic Modelling*, 19(1), 105-120.

  Integrates unemployment into the Solow model. Using panel data from 13 OECD countries between 1960 and 1990, they find that an increase in unemployment reduces long-term productivity levels.