ECN 710 : Advanced Macroeconomics

Chapter 3: The standard neoclassical model

Ramsey optimal growth model

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Introduction: Foundations of the Neoclassical Model

- In the Solow model, the savings rate is constant and exogenous.
- In the neoclassical growth model,
  - ☞ The preferences (utilities) of households are clearly specified.
  - ☞ Households choose consumption and investment optimally to maximize their utility.
- The neoclassical growth model is also called the Ramsey model or the Cass-Koopmans model.
- Difference with the Solow model:
  - ☞ The endogenous treatment of savings and labor supply.

Economic Environment: Preferences (1/5)

- Finite horizon and discrete time.
- The household derives utility from consumption and leisure.
- Let $c_t \equiv C_t / L_t$ and z_t be per capita consumption and leisure at time t.

  C_t: total consumption and L_t: population size.
- Preferences are defined over consumption and leisure paths, $\mathbf{x} = \{x_t\}_{t=0}^{\infty}$, with $x_t = (c_t, z_t)$, and are represented by the utility function

$$\begin{array}{rccc} \mathscr{U}: & \mathbb{X}^{\infty} & \to & \mathbb{R} \\ & x & \mapsto & \mathscr{U}(x_0, x_1, \ldots) \end{array} \quad (1)$$

  $\mathbb{X}$ is the domain of x_t and is typically $\mathbb{R}_+ \times [0, 1]$.

Economic Environment: Preferences (2/5)

- Preferences are said to be recursive if there exists a function $W : \mathbb{X} \times \mathbb{R} \to \mathbb{R}$ (often called a utility aggregator) such that, for any $\{x_t\}_{t=0}^{\infty}$,

$$\mathscr{U}(x_0, x_1, \ldots) = W[x_0, \mathscr{U}(x_1, x_2, \ldots)]$$

- Thus, each $\{x_t\}_{t=0}^{\infty}$ induces a utility path $\{\mathscr{U}_t\}_{t=0}^{\infty}$ according to the following recursion

$$\mathscr{U}_t = W(x_t, \mathscr{U}_{t+1})$$

- Preferences are additively separable if there exist functions v_t such that

$$\mathscr{U}(\mathbf{x}) = \sum_{t=0}^{\infty} v_t(x_t).$$

- $v_t(x_t)$: the utility at period 0 from consumption and leisure at period t.

Economic Environment: Preferences (3/5)

- We will assume that preferences are recursive and additively separable.

  $\Longrightarrow$ The utility aggregator W must be linear in its second argument:

  $\Longrightarrow$ There exists a function $U : \mathbb{R} \to \mathbb{R}$ and a scalar $\beta \in \mathbb{R}$ such that

$$W(x, y) = U(x) + \beta y.$$

  This implies that

$$\mathscr{U}_t = U(x_t) + \beta \mathscr{U}_{t+1}$$

  or equivalently,

$$\mathscr{U}_t = \sum_{\tau=0}^{\infty} \beta^{\tau} U(x_{t+\tau})$$

- $\beta \in (0, 1)$ is the discount factor and U is called the instantaneous utility function.

- We assume that the maximum amount of time per period is 1. Thus,

$$\mathbb{X} = \mathbb{R}_+ \times [0, 1].$$

- Special case $t = 0$

$$\mathscr{U}_0 = \sum_{\tau=0}^{\infty} \beta^{\tau} U(x_{\tau}) = \sum_{\tau=0}^{\infty} \beta^{\tau} U(c_{\tau}, z_{\tau})$$

- $\mathscr{U}_0$ is the function that the household seeks to optimize by choosing a path of consumption and leisure $\{c_t, z_t\}_{t=0}^{\infty}$.

Economic Environment: Neoclassical Preferences (5/5)

Assumption 1

The function U must be neoclassical, i.e.:

- Continuous and twice differentiable.
- Strictly increasing and strictly concave

$$U_c(c,z) > 0 > U_{cc}(c,z)$$

$$U_z(c,z) > 0 > U_{zz}(c,z)$$

$$U_{cz}^2 < U_{cc}U_{zz}.$$

- Satisfies the Inada conditions:

$$\lim_{c\to 0} U_c = \infty, \quad \lim_{c\to\infty} U_c = 0, \quad \lim_{z\to 0} U_z = \infty \quad \text{and} \quad \lim_{z\to 1} U_z = 0. \tag{2}$$

- All firms have access to the same production technology:

  $\implies$ The economy has a representative firm.

- Factor and product markets are competitive.

- The aggregate production function for the single final good is

$$Y(t) = F(K_t, L_t, A_t) \tag{3}$$

- K_t and L_t: demand for capital and labor at time t.

- A_t is the technology at time t.

Assumption 2

(Continuity, Differentiability, Diminishing and Positive Marginal Products, and Constant Returns to Scale).

The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ is twice differentiable in K and L, and satisfies:

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0$$
$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0 \tag{4}$$

Furthermore, F has constant returns to scale in K and L.

Economic Environment: The Firm (2/4)

Assumption 3:

(**Inada Conditions**).

The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfies the Inada conditions:

$$\begin{aligned} \lim_{K \to 0} F_K(K, L, A) = \infty \quad \text{and} \quad \lim_{K \to \infty} F_K(K, L, A) = 0, \quad \forall L > 0 \\ \lim_{L \to 0} F_L(K, L, A) = \infty \quad \text{and} \quad \lim_{L \to \infty} F_L(K, L, A) = 0, \quad \forall K > 0 \end{aligned} \tag{5}$$

$\Longrightarrow$ Ensure the existence of interior equilibria;

$\Longrightarrow$ All factors of production are necessary, i.e.

$$F(0, L, A) = F(K, 0, A) = 0. \tag{6}$$

Economic Environment: The Firm (3/4)

- A production function that satisfies assumptions (2) and (3) is called a **neoclassical production function or technology**.
- In intensive notation (i.e., the production function per worker):

$$y_t \equiv \frac{Y_t}{L} = F\left(\frac{K_t}{L}, \frac{L_t}{L}, A_t\right) \equiv F\left(k_t, \ell_t\right)$$

where A_t is assumed constant (equal to 1), $k_t \equiv K_t/L$, and $\ell_t \equiv L_t/L$.

- Factor prices:

$$R_t = F_k\left(k_t, \ell_t\right) \tag{7}$$

$$w_t = F_L\left(k_t, \ell_t\right) \tag{8}$$

Economic Environment: The Firm (4/4)

- The time constraint is given by:

$$\ell_t + z_t \leq 1. \tag{9}$$

z_t and ℓ_t are interpreted as the fractions of time the household spends on leisure and work, respectively.

- Since the economy is closed, the resource constraint (per worker) is given by:

$$c_t + i_t \leq y_t. \tag{10}$$

where i_t is the investment per worker at time t.

Technology and Resource Constraint

- The law of motion for the capital stock is:

$$k_{t+1} = i_t + (1-\delta)k_t. \tag{11}$$

- By combining equations (10) and (11), we obtain:

$$c_t + k_{t+1} \leq F(k_t, \ell_t) + (1-\delta)k_t \tag{12}$$

- Finally, we impose the following natural non-negativity constraints:

$$c_t \geq 0, \quad \ell_t \geq 0, \quad z_t \geq 0, \quad k_t \geq 0.$$

- In equilibrium, the time constraint will be saturated, which implies:

$$\ell_t = 1 - z_t. \tag{13}$$

Social Planner's Problem (1/3)

- We begin the analysis of the neoclassical growth model by considering the optimal allocation of a benevolent social planner.
- The SP chooses the static and intertemporal allocation of resources in the economy to maximize social welfare.
- We will later determine the allocations in a decentralized **competitive market environment.**
- We will show that the two allocations coincide.

Social Planner's Problem (2/3)

- The SP chooses a path $\{c_t, \ell_t, k_{t+1}\}_{t=0}^{\infty}$ that maximizes utility subject to the economy's resource constraint, with a given initial $k_0 > 0$:

$$\max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \mathscr{U}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t)$$

$$c_t + k_{t+1} \leq (1-\delta)k_t + F(k_t, \ell_t), \quad \forall t \geq 0,$$

$$c_t \geq 0, \quad \ell_t \in [0,1], \quad k_{t+1} \geq 0, \quad \forall t \geq 0,$$

$$k_0 > 0 \text{ given.}$$

- The household will never choose $c_t = 0$, $k_{t+1} = 0$, $\ell_t = 0$, or $\ell_t = 1$. **Why**?

$\Longrightarrow c_t \geq 0, \quad \ell_t \in [0,1], \quad k_{t+1} \geq 0$ will often be ignored in the solution.

Social Planner's Problem (3/3)

- Let μ_t denote the Lagrange multiplier for the resource constraint.
- The Lagrangian for the social planner's problem is written as follows:

$$\mathscr{L}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t) + \sum_{t=0}^{\infty} \mu_t \left[(1-\delta)k_t + F(k_t,\ell_t) - k_{t+1} - c_t\right]$$

- The first-order conditions (FOCs) are:

$$\frac{\partial \mathscr{L}_0}{\partial c_t} = \beta^t U_c(c_t, 1-\ell_t) - \mu_t = 0, \quad (14)$$

$$\frac{\partial \mathscr{L}_0}{\partial \ell_t} = -\beta^t U_z(c_t, 1-\ell_t) + \mu_t F_L(k_t,\ell_t) = 0, \quad (15)$$

$$\frac{\partial \mathscr{L}_0}{\partial k_{t+1}} = -\mu_t + \mu_{t+1}\left[(1-\delta) + F_K(k_{t+1},\ell_{t+1})\right] = 0. \quad (16)$$

Characterization of Equilibrium (1/2)

- By combining the above, we obtain:

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = F_L(k_t, \ell_t), \tag{17}$$

$$\frac{U_c(c_t, 1-\ell_t)}{\beta U_c(c_{t+1}, 1-\ell_{t+1})} = 1 - \delta + F_K(k_{t+1}, \ell_{t+1}). \tag{18}$$

- Equation (17) means that the marginal rate of substitution between consumption and leisure equals the marginal product of labor.
- Equation (18) equates the intertemporal marginal rate of substitution in consumption to the net marginal product of capital (including depreciation).
- The latter condition is called the **Euler equation**.

Characterization of Equilibrium (2/2)

- Interpretation of equation (17):

$$\underbrace{U_z(c_t, 1-\ell_t)}_{\substack{\text{Disutility from working} \\ \text{one unit of time}}} = \underbrace{F_L(k_t, \ell_t)}_{\substack{\text{Income from} \\ \text{one unit of work}}} \times \underbrace{U_c(c_t, 1-\ell_t)}_{\substack{\text{Utility from} \\ \text{consuming \$1}}} . \tag{19}$$

- Interpretation of the Euler equation (18):

$$\underbrace{U_c(c_t, 1-\ell_t)}_{\substack{\text{Utility lost} \\ \text{by saving \$1}}} = \underbrace{[1-\delta+F_K(k_{t+1}, \ell_{t+1})]}_{\substack{\text{Return in } t+1 \\ \text{on \$1 invested in } t}} \times \underbrace{\beta U_c(c_{t+1}, 1-\ell_{t+1})}_{\substack{\text{Utility of consuming} \\ \text{\$1 in } t+1}} . \tag{20}$$

☞ Impatience: If β decreases, current consumption increases, and future consumption decreases.

☞ Return on investment: If it increases, more saving occurs for greater future consumption.

Social Planner's Problem: Finite Horizon 1/6

Suppose the horizon is finite, $T < \infty$.

- The social planner's problem can be written as:

$$\max_{\{c_t, \ell_t, k_{t+1}\}_{t=0}^{T}} \mathscr{U}_0 = \sum_{t=0}^{T} \beta^t U(c_t, 1 - \ell_t)$$

$$c_t + k_{t+1} \leq (1-\delta)k_t + F(k_t, \ell_t), \quad \forall t = 0, \cdots, T,$$

$$c_t \geq 0, \quad \ell_t \in [0,1], \quad k_{t+1} \geq 0, \quad \forall t = 0, \cdots, T,$$

$$k_0 > 0 \text{ given.}$$

- The planner will never choose $c_t = 0$, $\ell_t = 0$, or $\ell_t = 1$, $\forall t = 0, \cdots, T$.

Social Planner's Problem: Finite Horizon 2/6

- For $k_{t+1} \geq 0$, the decision becomes non-trivial due to the choice at T.
- We need to introduce a multiplier λ_t for this constraint.
- The Lagrangian for the social planner's problem is:

$$\mathscr{L}_0 = \sum_{t=0}^{T} \beta^t U(c_t, 1-\ell_t) + \sum_{t=0}^{T} \mu_t [(1-\delta)k_t + F(k_t, \ell_t) - k_{t+1} - c_t] + \sum_{t=0}^{T} \lambda_t k_{t+1}.$$

- We calculate the FOCs and solve the problem.
- We can also use the **Kuhn-Tucker conditions** for optimization with non-negativity constraints.

Social Planner's Problem: Finite Horizon (3/6)

Kuhn-Tucker Theorem

Assume x^* maximizes the following problem:

$$\max_{x \in \mathbb{R}^n} f(x)$$

$$\text{s.t. } g_1(x) = b_1, \ldots, g_M(x) = b_M,$$

$$h_1(x) \leq c_1, \ldots, h_K(x) \leq c_K.$$

- This is a constrained maximization problem.
  - M equality constraints,
  - K inequality constraints.
- Assume the constraint qualification condition is satisfied at x^*.

- Form the Lagrangian:

$$L = f(x) + \sum_{m=1}^{M} \lambda_m (b_m - g_m(x)) + \sum_{k=1}^{K} \mu_k (c_k - h_k(x))$$

- First-order conditions: find x^* such that

$$\frac{\partial L(x^*)}{\partial x_n} = \frac{\partial f(x^*)}{\partial x_n} - \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x^*)}{\partial x_n} - \sum_{k=1}^{K} \mu_k \frac{\partial h_k(x^*)}{\partial x_n} = 0,$$

for all $n = 1, \ldots, N$, and

$$h_k(x) \leq c_k, \quad \mu_k \geq 0, \quad \text{and} \quad \mu_k (c_k - h_k(x^*)) = 0,$$

for all $k = 1, \ldots, K$.

- Return to the Social Planner's (SP) problem. The Kuhn-Tucker conditions with respect to k_{T+1} are written as:

$$\frac{\partial \mathscr{L}_0}{\partial k_{T+1}} \geq 0, \quad k_{T+1} \geq 0, \quad \text{and} \quad \frac{\partial \mathscr{L}_0}{\partial k_{T+1}} k_{T+1} = 0 \tag{21}$$
$$\implies \quad \lambda_T \geq 0, \quad k_{T+1} \geq 0, \quad \text{and} \quad \lambda_T k_{T+1} = 0.$$

- This implies that $k_{T+1} = 0$, meaning the shadow value of k_{T+1} is zero.
- When $T = \infty$, the terminal condition $\mu_T k_{T+1} = 0$ is replaced by the **transversality condition**:

$$\lim_{t \to \infty} \mu_t k_{t+1} = 0. \tag{22}$$

- This means that the discounted shadow value of capital converges to zero:

$$\lim_{t \to \infty} \beta^t U_c(c_t, \ell_t) k_{t+1} = 0. \tag{23}$$

Social Planner's Problem: Finite Horizon (6/6)

Proposition

The path $\{c_t, \ell_t, k_{t+1}\}_{t=0}^{\infty}$ is a solution to the social planner's problem if and only if the following conditions hold for all $t \geq 0$:

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = F_L(k_t, \ell_t),$$

$$\frac{U_c(c_t, 1-\ell_t)}{\beta U_c(c_{t+1}, 1-\ell_{t+1})} = 1-\delta+F_K(k_{t+1}, \ell_{t+1}),$$

$$k_{t+1} = F(k_t, \ell_t) + (1-\delta)k_t - c_t.$$

The initial condition is

$$k_0 > 0 \quad \text{(given)}.$$

The transversality condition is

$$\lim_{t\to\infty} \beta^t U_c(c_t, 1-\ell_t)\, k_{t+1} = 0.$$

Application (1/2)

Consider the following utility function:

$$u(c_t, \ell_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \frac{\ell_t^{\gamma}}{1+\gamma}.$$

- u is additively separable in consumption and leisure.
- The intertemporal marginal rate of substitution of consumption is

$$MRS = \frac{\beta u'(c_{t+1})}{u'(c_t)}.$$

- The elasticity of intertemporal substitution in consumption is

$$\frac{\partial (c_{t+1}/c_t)}{\partial MRS} \cdot \frac{MRS}{c_{t+1}/c_t} = \frac{1}{\sigma}.$$

Consider the following neoclassical production function:

$$Y_t = F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha},$$

- where $0 < \alpha < 1$.
- F is a Cobb-Douglas production function.
- Output per worker is given by

$$y_t = \frac{Y_t}{L_t} = Ak_t^{\alpha} \ell^{1-\alpha}.$$

- **Formulate the Social Planner's problem and solve it to derive the Euler equation in finite and infinite horizon.**