ECN 710 : Advanced Macroeconomics

Chapter 4: The standard neoclassical model

Competitive equilibrium markets

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Introduction

- In the optimal equilibrium, the social planner decides on allocations in the economy (Lecture 3).
- Whereas a competitive equilibrium is a vector of prices and quantities such that:
  - ☞ Households choose quantities that maximize their utility given their budget constraint; they take prices as given.
  - ☞ Firms choose the level of production (and the quantities of inputs) that maximize their profit; they take prices as given.
  - ☞ Markets are in equilibrium (prices are such that supply equals demand in all markets).

Household Preferences

- We consider a representative household.
- We assume there is no population growth.
- The household's preferences are given by

$$\mathscr{U}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, z_t) \tag{1}$$

- In recursive form, $\mathscr{U}_t = U(c_t, z_t) + \beta \mathscr{U}_{t+1}$.
- The household's time constraint can be written as follows

$$z_t = 1 - \ell_t \tag{2}$$

Household Budget Constraint (1/2)

- The household's budget constraint is given by

$$c_t + i_t + x_t \leq r_t k_t + R_t b_t + w_t \ell_t + \alpha \Pi_t. \tag{3}$$

  - r_t denotes the rental rate of capital,
  - w_t the wage rate,
  - R_t the interest rate on risk-free bonds,
  - α the share of profit Π_t paid to the household,
  - x_t the investment in bonds.

Household Budget Constraint (2/2)

- The household accumulates capital according to the following law of motion

$$k_{t+1} = (1-\delta)k_t + i_t \tag{4}$$

and bonds according to

$$b_{t+1} = b_t + x_t \tag{5}$$

- In equilibrium, firm profits are zero due to perfect competition. It follows that $\Pi_t = 0$.
- The budget constraint (3) is rewritten as follows:

$$c_t + k_{t+1} + b_{t+1} \leq (1-\delta+r_t)\,k_t + (1+R_t)\,b_t + w_t\ell_t. \tag{6}$$

Household Debt Limit (1/3)

- The non-negativity constraint

$$k_{t+1} \geq 0 \tag{7}$$

- No sign constraint is imposed on b_t.
- The household can either lend or borrow bonds.
- We only impose the following natural borrowing constraint or ”No Ponzi Game” condition:

$$-(1+R_{t+1})\, b_{t+1} \leq (1-\delta+r_{t+1})\, k_{t+1} + \sum_{\tau=t+1}^{\infty} \frac{q_\tau}{q_{t+1}} w_\tau \tag{8}$$

with

$$q_t \equiv \frac{1}{(1+R_0)(1+R_1)\ldots(1+R_t)} = (1+R_{t+1})\, q_{t+1}.$$

- ”No Ponzi Game” condition requires that the household’s net debt does not exceed the present value of the income it can earn by working all the time.
- The arbitrage between bonds and capital implies that in equilibrium:

$$R_t = r_t - \delta \tag{9}$$

- If $R_t < r_t - \delta$, all individuals would want to short-sell bonds, and there would be an excess supply of bonds.
- If $R_t > r_t - \delta$, no one in the economy would invest in capital.
- The household is then indifferent between bonds and capital.

- If we consider that $a_t = b_t + k_t$ represents the total assets, the budget constraint (6) reduces to

$$c_t + a_{t+1} \leq (1 + R_t)\, a_t + w_t \ell_t \tag{10}$$

- and the natural borrowing constraint becomes $a_{t+1} \geq \underline{a}_{t+1}$, where

$$\underline{a}_{t+1} \equiv -\frac{1}{q_t} \sum_{\tau=t+1}^{\infty} q_\tau w_\tau = -\sum_{\tau=t+1}^{\infty} \left[\prod_{j=t+1}^{\tau} \frac{1}{1+R_j} \right] w_\tau \tag{11}$$

- We assume $\underline{a}_t$ is bounded. i.e. prices $\{R_t, w_t\}_{t=0}^{\infty}$ are such that:

$$\frac{1}{q_t} \sum_{\tau=t+1}^{\infty} q_\tau w_\tau < \infty.$$

Household Problem (1/9)

- Given a sequence of prices $\{R_t, w_t\}_{t=0}^{\infty}$, the household chooses a sequence of $\{c_t, \ell_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize lifetime utility subject to its budget constraints.

$$\max_{\{c_t,\ell_t,a_{t+1}\}_{t=0}^{\infty}} \quad \mathscr{U}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t)$$

$$\text{s.t.} \quad c_t + a_{t+1} \leq (1+R_t)\, a_t + w_t \ell_t, \quad \forall t$$

$$c_t \geq 0, \quad \ell_t \in [0,1], \quad a_{t+1} \geq \underline{a}_{t+1}, \quad \forall t$$

- If $\mu_t = \beta^t \lambda_t$ is the Lagrange multiplier for the budget constraint, we can write the Lagrangian as follows

$$\mathscr{L}_0 \quad = \quad \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t, 1-\ell_t) + \lambda_t \left[(1+R_t)\, a_t + w_t \ell_t - a_{t+1} - c_t \right] \right\}$$

- The FOC with respect to c_t is

$$\frac{\partial \mathscr{L}_0}{\partial c_t} = 0 \quad \Leftrightarrow \quad U_c\left(c_t, z_t\right) = \lambda_t$$

- The FOC with respect to ℓ_t is

$$\frac{\partial \mathscr{L}_0}{\partial \ell_t} = 0 \quad \Leftrightarrow \quad U_z\left(c_t, z_t\right) = \lambda_t w_t$$

- These first two FOCs imply that

$$\frac{U_z\left(c_t, z_t\right)}{U_c\left(c_t, z_t\right)} = w_t$$

- Households equate their marginal rate of substitution between consumption and leisure with the (common) wage rate.

- The Kuhn-Tucker conditions with respect to a_{t+1} are written as

$$\frac{\partial \mathscr{L}_0}{\partial a_{t+1}} = \beta^t \left[-\lambda_t + \beta \left(1 + R_{t+1}\right) \lambda_{t+1}\right] \leq 0, \tag{12}$$

and

$$a_{t+1} \geq \underline{a}_{t+1}, \quad \left[\lambda_t - \beta \left(1 + R_{t+1}\right) \lambda_{t+1}\right] \left[a_{t+1} - \underline{a}_{t+1}\right] = 0. \tag{13}$$

- Using $\lambda_t = U_c\left(c_t, z_t\right)$, the Euler condition (12) becomes

$$U_c\left(c_t, z_t\right) \geq \beta \left(1 + R_{t+1}\right) U_c\left(c_{t+1}, z_{t+1}\right) \tag{14}$$

Household Problem (4/9)

- If $a_{t+1} > \underline{a}_{t+1}$ then $U_c(c_t, z_t) = \beta(1 + R_{t+1}) U_c(c_{t+1}, z_{t+1})$

$\Rightarrow$ When the borrowing constraint is not binding, households equate their intertemporal marginal rate of substitution with the (common) return on capital.

- If $a_{t+1} = \underline{a}_{t+1}$ then $U_c(c_t, z_t) > \beta(1 + R_{t+1}) U_c(c_{t+1}, z_{t+1})$
- In this case, $c_\tau = z_\tau = 0$ for all $\tau \geq t$, then we have

$$U_c(c_{t+1}, z_{t+1}) = \infty \Rightarrow U_c(c_t, z_t) > \infty,$$

- Which is absurd given that $U_c(c_t, z_t) < \infty$.
- This shows that the borrowing constraint can never be binding.

The finite horizon Lagrangian is written as

$$\mathscr{L}_0 = \sum_{t=0}^{T} \beta^t \{U(c_t, 1-\ell_t) + \lambda_t [(1+R_t) a_t + w_t \ell_t - a_{t+1} - c_t]\}$$

- The FOC with respect to a_{t+1} for $t = 0, 1 \cdots T-1$ is

$$-\lambda_t + \beta (1 + R_{t+1}) \lambda_{t+1} \leq 0, \tag{15}$$

- The Kuhn-Tucker condition with respect to a_{T+1} is written as

$$\lambda_T \geq 0, \qquad a_{T+1} \geq \underline{a}_{T+1}, \quad \lambda_T \left[a_{T+1} - \underline{a}_{T+1}\right] = 0. \tag{16}$$

- By multiplying by β^T and letting T tend to infinity, we obtain

$$\lim_{T \to \infty} \beta^T \lambda_T \left[a_{T+1} - \underline{a}_{T+1}\right] = 0. \tag{17}$$

Household Problem (6/9)

- If the borrowing constraint is never binding, equation (15) is written as

$$\lambda_t = \beta \left[1 + R_{t+1}\right] \lambda_{t+1}.$$

- This implies that

$$\beta^t \lambda_t = \prod_{\tau=1}^{t} \frac{1}{1+R_\tau} \lambda_0 = (1+R_0) q_t \lambda_0. \tag{18}$$

- We can then rewrite the terminal condition (17) as follows

$$\lim_{t\to\infty} \beta^t \lambda_t a_{t+1} = \lim_{t\to\infty} \beta^t \lambda_t \underline{a}_{t+1} = (1+R_0)\lambda_0 \lim_{t\to\infty} q_t \underline{a}_{t+1}. \tag{19}$$

- But note that

$$q_t \underline{a}_{t+1} = \sum_{\tau=t}^{\infty} q_\tau w_\tau \tag{20}$$

Household Problem (7/9)

- Recall that

$$q_t \underline{a}_{t+1} = \sum_{\tau=t}^{\infty} q_\tau w_\tau, \quad and \quad \sum_{\tau=0}^{\infty} q_\tau w_\tau < \infty.$$

- Then,

$$\lim_{t\to\infty} \sum_{\tau=t}^{\infty} q_\tau w_\tau = 0.$$

- We obtain the more familiar version of the transversality condition

$$\lim_{t\to\infty} \beta^t \lambda_t a_{t+1} = 0$$

- Which is equivalently written as

$$\lim_{t\to\infty} \beta^t U_c\left(c_t, 1-\ell_t\right) a_{t+1} = 0. \tag{21}$$

- This allows us to reformulate the household problem in the manner of **Arrow-Debreu**:

$$\max_{\{c_t,\ell_t,k_{t+1},b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t)$$

$$\text{s.t.} \sum_{t=0}^{\infty} q_t c_t \leq a_0 + \sum_{t=0}^{\infty} q_t w_t \ell_t$$

with

$$a_0 + \sum_{t=0}^{\infty} q_t w_t < \infty.$$

- The intertemporal budget constraint is equivalent to the sequence of period-by-period budget constraints and the natural borrowing limit as written on slide 11.

- Let $\mu > 0$ be the Lagrange multiplier associated with the intertemporal budget.
- The FOCs with respect to c_t and ℓ_t give

$$\beta^t U_c(c_t, 1-\ell_t) = \mu q_t$$

and

$$\beta^t U_z(c_t, 1-\ell_t) = \mu q_t w_t,$$

- We can verify that these conditions coincide with those derived previously.

Definition

Suppose the sequence of prices $\{R_t, r_t, w_t\}_{t=0}^{\infty}$ satisfies

$$R_t = r_t - \delta \text{ for all } t, \quad \sum_{t=0}^{\infty} q_t < \infty \quad and \quad \sum_{t=0}^{\infty} q_t w_t < \infty$$

The sequence $\{c_t, \ell_t, a_t\}_{t=0}^{\infty}$ solves the individual household problem if and only if

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = w_t, \quad \frac{U_c(c_t, 1-\ell_t)}{\beta U_c(c_{t+1}, 1-\ell_{t+1})} = 1 + R_t, \; c_t + a_{t+1} = (1+R_t)\, a_t + w_t \ell_t, \quad \forall\, t$$

$$\text{with} \quad a_0 > 0 \text{ given and } \lim_{t \to \infty} \beta^t U_c(c_t, 1-\ell_t)\, a_{t+1} = 0$$

Given $\{a_t\}_{t=1}^{\infty}$, an optimal portfolio is any $\{k_t, b_t\}_{t=1}^{\infty}$ such that $k_t \geq 0$ and $b_t = a_t - k_t$. Recall that leisure $z_t = 1 - \ell_t$

Firms (1/2)

- We assume there is a representative firm
- The representative firm employs labor and rents capital in labor and capital markets.
- The firm has access to the same technology and produces a homogeneous good that it sells competitively to households.
- Let K_t and L_t be the quantities of capital and labor that the firm employs at time t.
- The firm seeks to maximize its profit at time t:

$$\max_{\{K_t, L_t\}} \Pi_t = F(K_t, L_t) - r_t K_t - w_t L_t \tag{22}$$

Firms (2/2)

- The first-order conditions are written as

$$F_K(K_t, L_t) = r_t, \quad and \quad F_L(K_t, L_t) = w_t. \tag{23}$$

- They imply the capital-labor ratio of each firm (K_t/L_t), but not the size of the firm (L_t).
- An interior solution to the firms' problem exists if and only if r_t and w_t imply the same K_t/L_t.
- Since all firms have access to the same technology, they use exactly the same capital-labor ratio.
- Given that the function F has constant returns to scale, profit is zero in equilibrium:

$$\Pi_t = 0.$$

1. Introduction
2. Households
3. Firms
4. **Competitive Equilibrium**
5. Optimal Equilibrium vs Competitive Equilibrium
6. Recursive Competitive Equilibrium

Market Equilibrium

- The bond market is in equilibrium at date t if and only if

$$0 = b_t, \tag{24}$$

  with $b_t = a_t - k_t$.

- The capital market is in equilibrium at date t if and only if

$$K_t = k_t. \tag{25}$$

- The labor market is in equilibrium at date t if and only if

$$L_t = \ell_t. \tag{26}$$

Definition of Competitive Equilibrium: Arrow-Debreu Formulation

A (competitive) equilibrium of the economy is a sequence of allocations $\{c_t, \ell_t, k_{t+1}, b_{t+1}, K_t, L_t\}_{t=0}^{\infty}$ and prices $\{p_t, R_t, r_t, w_t\}_{t=0}^{\infty}$ such that

(i) Given $\{p_t, R_t, r_t, w_t\}_{t=0}^{\infty}$, the path $\{c_t, \ell_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}$ solves the household problem:

$$\max_{\{c_t, \ell_t, a_{t+1}\}_{t=0}^{\infty}} \quad \mathscr{U}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - \ell_t)$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t [c_t + a_{t+1}] \leq \sum_{t=0}^{\infty} p_t [(1 + R_t) a_t + w_t \ell_t] \quad \text{with} \;\; a_{t+1} \geq \underline{a}_{t+1} \; \forall t$$

(ii) Given (r_t, w_t), the pair (K_t, L_t) maximizes the firm's profit for each t.
$\max_{\{K_t, L_t\}} \Pi_t = F(K_t, L_t) - r_t K_t - w_t L_t$

(iii) The bond, capital, and labor markets are in equilibrium at each period, i.e., equations (24), (25), and (26) are satisfied for each date.

Definition of Competitive Equilibrium: Sequential Formulation

A (competitive) equilibrium of the economy is a sequence of allocations $\left\{(c_t, \ell_t, k_{t+1}, b_{t+1})_{j\in[0,L_t]}, (K_t, L_t)\right\}_{t=0}^{\infty}$ and prices $\{p_t, R_t, r_t, w_t\}_{t=0}^{\infty}$ such that

(i) Given $\{R_t, r_t, w_t\}_{t=0}^{\infty}$, the path $\{c_t, \ell_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}$ solves the household problem:

$$\max_{\{c_t,\ell_t,a_{t+1}\}_{t=0}^{\infty}} \quad \mathscr{U}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t)$$

$$\text{s.t.} \quad c_t + a_{t+1} \le (1+R_t)a_t + w_t\ell_t \quad \forall t \quad \text{with} \quad \lim_{T\to\infty} q_t a_{t+1} = 0.$$

(ii) Given (r_t, w_t), the pair (K_t, L_t) maximizes the firm's profit for each t.
$\max_{\{K_t,L_t\}} \Pi_t = F(K_t, L_t) - r_t K_t - w_t L_t$

(iii) The bond, capital, and labor markets are in equilibrium at each period, i.e., equations (24), (25), and (26) are satisfied for each date.

Optimal Equilibrium vs Competitive Equilibrium

- The optimal equilibrium refers to the equilibrium of the social planner's problem.

Proposition

The set of competitive equilibrium allocations for the market economy coincides with the set of optimal allocations of the social planner.

Optimal Equilibrium ⇒ Competitive Equilibrium

We first examine how the solution to the social planner's problem can correspond to the solution of a competitive equilibrium.

The social planner's optimal plan is a sequence of allocations $\{c_t, \ell_t, k_t\}_{t=0}^{\infty}$ such that

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = F_L(k_t, \ell_t), \quad \forall t \geq 0, \tag{27}$$

$$\frac{U_c(c_t, 1-\ell_t)}{\beta U_c(c_{t+1}, 1-\ell_{t+1})} = [1-\delta + F_K(k_{t+1}, \ell_{t+1})], \quad \forall t \geq 0 \tag{28}$$

$$c_t + k_{t+1} = (1-\delta)k_t + F(k_t, \ell_t), \quad \forall t \geq 0 \tag{29}$$

$$k_0 > 0 \text{ given, and } \lim_{t \to \infty} \beta^t U_c(c_t, 1-\ell_t) k_{t+1} = 0. \tag{30}$$

Let the price path $\{R_t, r_t, w_t\}_{t=0}^{\infty}$ be given by

$$r_t = F_K(k_t, \ell_t), \tag{31}$$

$$R_t = r_t - \delta, \tag{32}$$

$$w_t = F_L(k_t, \ell_t). \tag{33}$$

For each household and each firm, define the allocations

$$c_t = c_t, \quad \ell_t = \ell_t \quad \text{and} \quad K_t/L_t = k_t, \qquad \forall t.$$

Equations (28), (31), and (32) imply

$$\frac{U_c(c_t, 1-\ell_t)}{\beta U_c(c_{t+1}, 1-\ell_{t+1})} = 1 + R_t. \tag{34}$$

Corollary

- The equilibrium is then given by an allocation $\{c_t, \ell_t, k_t\}_{t=0}^{\infty}$ such that, for all $t \geq 0$,

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = F_L(k_t, \ell_t), \tag{35}$$

$$\frac{U_c(c_t, 1-\ell_t)}{U_c(c_{t+1}, 1-\ell_{t+1})} = \beta\left[1-\delta+F_K(k_{t+1}, \ell_{t+1})\right], \tag{36}$$

$$k_{t+1} = F(k_t, \ell_t) + (1-\delta)k_t - c_t, \tag{37}$$

with $k_0 > 0$ and $\lim_{t\to\infty} \beta^t U_c(c_t, 1-\ell_t)\, k_{t+1} = 0$. Finally, the equilibrium prices are given by

$$R_t = F_K(k_t, \ell_t) - \delta, \quad r_t \equiv F_K(k_t, \ell_t), \quad w_t = F_L(k_t, \ell_t).$$

Recursive Competitive Equilibrium (1/10)

- Recursivity: intertemporal maximization is divided into decisions affecting the present and the future (through state variables).
- Instead of sequences, a recursive competitive equilibrium is a set of functions:
  - quantities
  - values
  - prices
- These functions describe the agents' choices and prices for given initial conditions.

- Consider again the social planner's problem. For any $k_0 > 0$, define

$$V(k_0) \equiv \max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t)$$

subject to the constraints

$$\begin{aligned} c_t + k_{t+1} &\leq (1-\delta)k_t + F(k_t,\ell_t), \quad \forall t \geq 0, \\ c_t, \ell_t, k_{t+1} &\geq 0, \quad \forall t \geq 0, \\ k_0 &> 0 \text{ given.} \end{aligned}$$

- V is called the value function.

- The constraint being saturated at equilibrium, we can write

$$c_t = (1-\delta)k_t + F(k_t, \ell_t) - k_{t+1}, \quad \forall t \geq 0$$

- The value function is then written as

$$V(k_0) \equiv \max_{\{\ell_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t H(\ell_t, k_t, k_{t+1})$$

with

$$H(\ell_t, k_t, k_{t+1}) \equiv U\left[(1-\delta)k_t + F(k_t, \ell_t) - k_{t+1}, 1-\ell_t\right]$$

$$
\begin{aligned}
V(k_0) &= \max_{\{\ell_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t H(\ell_t, k_t, k_{t+1}) \\
&= \max_{\ell_0, k_1} \left\{ H(\ell_0, k_0, k_1) + \max_{\{\ell_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^t H(\ell_t, k_t, k_{t+1}) \right\} \\
&= \max_{\ell_0, k_1} \left\{ H(\ell_0, k_0, k_1) + \beta \max_{\{\ell_t, k_{t+1}\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t H(\ell_{t+1}, k_{t+1}, k_{t+2}) \right\} \\
&= \max_{\ell_0, k_1} \{ H(\ell_0, k_0, k_1) + \beta V(k_1) \}
\end{aligned}
$$

Thus,

$$
V(k_0) = \max_{\ell_0, k_1} \{ U(c_0, 1 - \ell_0) + \beta V(k_1) \}
$$

Recursive Competitive Equilibrium (5/10)

- So we can write in general

$$V(k_t) = \max\{U(c_t, 1-\ell_t) + \beta V(k_{t+1})\} \tag{38}$$

- The Bellman equation for the previous planner's problem is written as:

$$V(k) = \max\left\{U(c, 1-\ell) + \beta V\left(k'\right)\right\}$$

$$\text{s.t.} \quad c + k' \leq (1-\delta)k + F(k,\ell)$$

$$k' \geq 0, \quad c \in [0, F(k,\ell)], \quad \ell \in [0,1].$$

- This is a formulation of the problem in recursive form.
- Let $c(k)$, $\ell(k)$, and $k'(k)$ be the values of c, ℓ, and k' that maximize $V(k)$. These expressions are also called policy functions.

Recursive Competitive Equilibrium (6/10)

- Let's return to the decentralized problem. **We use the budget constraint instead of a resource constraint.**
- Prices are given by the sequential formulation: $\{r_t, w_t\}_{t=0}^{\infty}$ such that

$$R = R(\bar{K})$$
$$w = w(\bar{K})$$

  where $\bar{K}$ is the aggregate capital
- Budget constraint in the recursive problem:

$$c + K' = R(\bar{K})K + w(\bar{K})\ell$$

- 2 variables (states) determine the consumer's choice:
  1. Their capital K
  2. The aggregate capital $\bar{K}$, which determines prices

Recursive Competitive Equilibrium (7/10)

- The consumer must therefore predict the evolution of aggregate capital.
- This prediction must be rational: it corresponds to the true law of motion

$$\bar{K}' = G(\bar{K})$$

where G is the result of the economy's capital accumulation choices (i.e., the representative consumer in this case).

- The household problem in recursive form (Bellman Equation) is then written as

$$V(K,\bar{K}) = \max_{c,\ell,K'\geq 0} \left\{ u(c,1-\ell) + \beta V\left(K',\bar{K}'\right) \right\}$$

$$\text{s.t. } c + K' = R(\bar{K})K + w(\bar{K})\ell$$

$$\bar{K}' = G(\bar{K})$$

Definition: Recursive Competitive Equilibrium

A recursive competitive equilibrium is a set of functions:

- quantities: $G(\bar{K})$ and $g(K, \bar{K})$
- value: $V(K, \bar{K})$
- prices: $R(\bar{K})$ and $w(\bar{K})$ such that:

  1. $V(K, \bar{K})$ solves (1) and $g(K, \bar{K})$ is the associated decision function
  2. prices are determined competitively:
$$R(\bar{K}) = F_K(\bar{K}, L) + 1 - \delta$$
$$w(\bar{K}) = F_L(\bar{K}, L)$$
  3. ”consistency”
$$g(\bar{K}, \bar{K}) = G(\bar{K})$$

Recursive Competitive Equilibrium (9/10)

- The consistency condition $G(\bar{K}) = g(\bar{K}, \bar{K})$ means that the law of motion perceived by the agent is correct
- In an economy with a single agent, $K = \bar{K}$ implies $G(\bar{K}) = g(\bar{K}, \bar{K})$
- With L_t agents:

$$\bar{K} = \sum_{i=1}^{L_t} K_i$$

and

$$G(\bar{K}) = \sum_{i=1}^{N} g_i\left(K_i, \bar{K}\right)$$

- Are the markets in equilibrium? In other words, is the following identity respected?

$$c + K' = F(\bar{K}, 1) + (1 - \delta)\bar{K}$$

Recursive Competitive Equilibrium (10/10)

- The definition of equilibrium tells us that the budget constraint is respected:

$$c + K' = R(\bar{K})K + w(\bar{K})\ell$$

- Since all firm revenues go to the consumer, we have

$$\begin{aligned} c + K' &= F_K(\bar{K}, 1)K + (1-\delta)K + F_n(\bar{K}, 1) \\ &= F(\bar{K}, 1) + (1-\delta)K \end{aligned}$$

  (Euler's theorem.)
- Finally, $K = \bar{K}$ and $g(\bar{K}, \bar{K}) = G(\bar{K})$ imply

$$c + \bar{K}' = F(\bar{K}, 1) + (1-\delta)\bar{K}.$$

Characterization: Recursive Competitive Equilibrium (1/3)

- Let's show that the solution to the competitive equilibrium problem is equivalent to those of the previous equilibria.

- The Lagrangian for the social planner's problem written in recursive form is

$$\mathscr{L} = U(c, 1-\ell) + \beta V\left(k'\right) + \lambda\left[(1-\delta)k + F(k,\ell) - k' - c\right]$$

- The first-order conditions with respect to c, ℓ, and k' are

$$\frac{\partial \mathscr{L}}{\partial c} = 0 \Leftrightarrow U_c(c,z) = \lambda$$

$$\frac{\partial \mathscr{L}}{\partial \ell} = 0 \Leftrightarrow U_z(c,z) = \lambda F_L(k,\ell)$$

$$\frac{\partial \mathscr{L}}{\partial k'} = 0 \Leftrightarrow \lambda = \beta V_k\left(k'\right)$$

- The envelope condition is $V_k(k) = \frac{\partial \mathscr{L}}{\partial k} = \lambda\left[1 - \delta + F_K(k,\ell)\right]$

Characterization: Recursive Competitive Equilibrium (2/3)

- By combining the two, we conclude

$$\frac{U_z\left(c_t, 1-\ell_t\right)}{U_c\left(c_t, 1-\ell_t\right)} = F_L\left(k_t, \ell_t\right)$$

and

$$\frac{U_c\left(c_t, \ell_t\right)}{U_c\left(c_{t+1}, \ell_{t+1}\right)} = \beta\left[1-\delta+F_K\left(k_{t+1}, \ell_{t+1}\right)\right],$$

- which are the same conditions we derived with optimal control.

- By combining the two, we conclude

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = F_L(k_t, \ell_t)$$

and

$$\frac{U_c(c_t, \ell_t)}{U_c(c_{t+1}, \ell_{t+1})} = \beta\left[1 - \delta + F_K(k_{t+1}, \ell_{t+1})\right],$$

- which are the same conditions we derived with optimal control.

Conclusion

Give assigment here!