ECN 710 : Advanced Macroeconomics

Chapter 5: The Real Business Cycle model

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1 Introduction

Summary

- Major stylized facts (revisited)
- How modern Macro explains business cycles
- The Real Business Cycle model: baseline version
- Recitation: how to transform functions in levels into log differences
- Linearizing the model in the vicinity of the steady state
- Numerical simulation of the linearized model
- Readings

Major stylized facts (revisited)

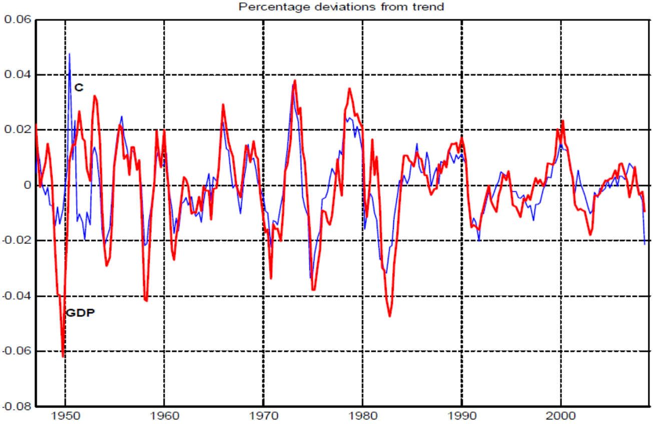

Figure: Percentage deviations from trend: GDP vs Consumption

Major stylized facts (revisited)

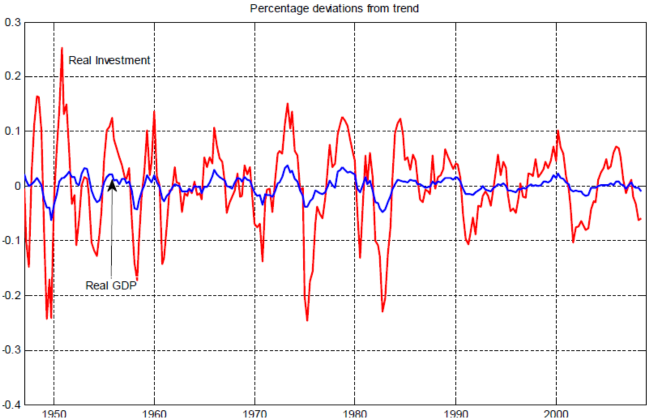

Figure: Percentage deviations from trend: GDP vs Investment

Major stylized facts (revisited)

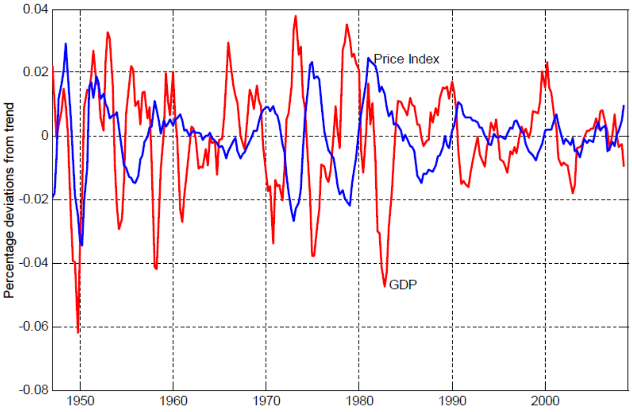

Figure: Percentage deviations from trend: GDP vs Price Index

Major stylized facts (revisited)

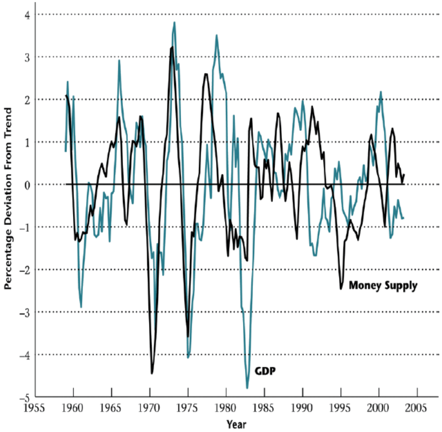

Figure: Percentage deviations from trend: GDP vs Money Supply

Major stylized facts (revisited)

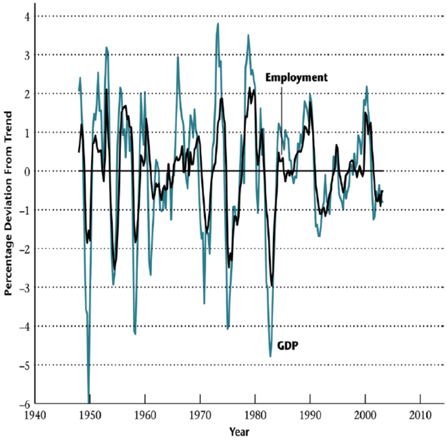

Figure: Percentage deviations from trend: GDP vs Employment

Major stylized facts (revisited)

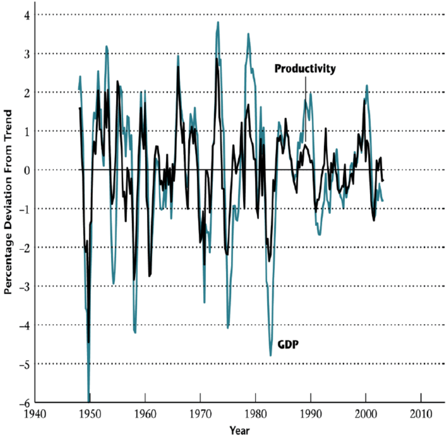

Figure: Percentage deviations from trend: GDP vs Productivity

Major stylized facts : Summary

From Stephen Williamson, Macroeconomics, Addison-Wesley, New York, 2005.

	Cyclicality	*Lead/Lag*	*Variability Relative to GDP*
Consumption	Procyclical	Coincident	Smaller
Investment	Procyclical	Coincident	Larger
Price Level	Countercyclical	Coincident	Smaller
Money Supply	Procyclical	Leading	Smaller
Employment	Procyclical	Lagging	Smaller
Real Wage	Procyclical	?	?
Average Labor Productivity	Procyclical	Coincident	Smaller

Figure: Percentage deviations from trend: GDP vs Productivity

Attention: price level as countercyclical and coincident is controversial!

How modern Macro explains business cycles

- Economies fluctuate over time
- Systematic facts needed to be explained:
  - Volatility
  - Comovements
  - Persistence (autocorrelation)
  - How expectations affect current economic decisions
- Dominant theoretical models:
  - Market clearing models: Real business cycles (RBC)
  - Non-Market clearing models: New Keynesian model (NKM)

RBC versus NKM

- A common framework:
  - Dynamic General Equilibrium
  - Stochastic shocks
  - Quantitative (computational)
  - Forward looking (Rational) Expectations
- A crucial divergence about information and prices:
  - Complete and flexible (RBC)
  - Incomplete and sticky (NKM)

The Real Business Cycle model: introduction

- The essence of the model:
  - Take the Solow growth model
  - Add shocks to Total Factor Productivity (TFP)
  - Add leisure to account for changes in hours of work
- Competitive equilibrium:
  - Households: preferences
  - Firms: technology
  - Government: policy decisions
- Real Factors: preferences, technology, policy decisions

TFP as the fundamental mechanism

- Shocks to Total Factor Productivity (TFP)
- Intertemporal substitution of labor and saving decisions
- Major result: fluctuations as an equilibrium outcome
  - Work harder when productivity is high
  - Save more when productivity is high

The Real Business Cycle model: baseline version

- Follow the baseline version by Hansen (1985)
- Seminal paper by Kydland and Prescott (1982)

Households: the problem

- Households maximize utility over time
- Utility depends on consumption (C) and hours worked (N)
- Intertemporal utility is discounted by a factor β

$$u() = \sum_{i=0}^{\infty} \beta^i u(C_{t+i}, N_{t+i}) \tag{1}$$

Households: with uncertainty

- Introducing uncertainty: future values of (C, N) are not known with certainty
- Expectations operator:

$$u() = E_t\left[\sum_{i=0}^{\infty}\beta^i u(C_{t+i}, N_{t+i})\right] \tag{2}$$

Households: utility function

- Specific form of utility:

$$u(C,N) = \frac{C^{1-\sigma}}{1-\sigma} - \theta N \tag{3}$$

$$\max E_t \left[\sum_{i=0}^{\infty} \beta^i \left(\frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \theta N_{t+i} \right) \right] \tag{4}$$

Firms: production

- Firms produce goods and services with the following production function:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{5}$$

Firms: accumulation of inputs

- Capital accumulation:

$$K_t = (1-\delta)K_{t-1} + I_t \tag{6}$$

- TFP:

$$\ln A_t = (1-\rho)\ln \bar{A} + \rho \ln A_{t-1} + \varepsilon_t \tag{7}$$

The social planner solution

- Solve for the equilibrium: decentralized and central planner equilibrium
- Social planner maximizes the objective function subject to a resource constraint

$$Y_t = C_t + I_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{8}$$

The Lagrangian

- The Lagrangian:

$$\mathscr{L} = E_t \sum_{i=0}^{\infty} \beta^i \left[\left(\frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \theta N_{t+i} \right) + \lambda_{t+i} \left(A_{t+i} K_{t+i-1}^{\alpha} N_{t+i}^{1-\alpha} + (1-\delta) K_{t+i-1} - C_{t+i} - K_{t+i} \right) \right] \tag{9}$$

The Lagrangian for two consecutive periods

- Simplify exposition: use $u(C_t, N_t)$ instead of $u = \frac{C_{t+i}^{1-\sigma}}{1-\sigma} - \theta N_{t+i}$
- The L function for t and $t+1$ is:

$$\begin{aligned}\mathscr{L} = \ldots + \beta^0 \left[u(C_t, N_t) + \lambda_t \left(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta)K_{t-1} - C_t - K_t\right)\right] \\ + \beta^1 \left[u(C_{t+1}, N_{t+1}) + \lambda_{t+1} \left(A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\delta)K_t - C_{t+1} - K_{t+1}\right)\right] \\ + \ldots \end{aligned} \tag{10}$$

- Now get the two first FOCs:

$$\frac{\partial L}{\partial C_t} = \beta^0 \left[u'_C(C_t) - \lambda_t\right] = 0 \tag{11}$$

$$\frac{\partial L}{\partial K_t} = \beta^0 \lambda_t + \beta^1 \lambda_{t+1} \left[\alpha \frac{A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha}}{K_t} + (1-\delta)\right] = 0 \tag{12}$$

The Lagrangian for two consecutive periods (cont.)

- Here is the $\mathscr{L}$ function for t and $t+1$ again:

$$\mathscr{L} = \ldots + \beta^0 \left[u(C_t, N_t) + \lambda_t \left(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) K_{t-1} - C_t - K_t \right) \right] + \beta^1 \left[u(C_{t+1}, N_{t+1}) + \lambda_{t+1} \left(A_{t+ \right. \right. \tag{13}$$

- Now let's go for the two last FOCs:

$$\frac{\partial \mathscr{L}}{\partial N_t} = \beta^0 \left[u'_N(N_t) + \lambda_t (1-\alpha) \frac{A_t K_{t-1}^{\alpha} N_t^{\alpha}}{N_t} \right] = 0 \tag{14}$$

$$\frac{\partial \mathscr{L}}{\partial \lambda_t} = \beta^0 \left[A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) K_{t-1} - C_t - K_t \right] = 0 \tag{15}$$

First Order Conditions (FOCs)

- The 4 FOCs can be written as:

$$\frac{\partial \mathscr{L}}{\partial C_t} = \beta^0 \left[u'_C(C_t) - \lambda_t\right] = 0 \quad (16)$$

$$\frac{\partial \mathscr{L}}{\partial K_t} = \beta^0 \lambda_t + \beta^1 \lambda_{t+1} \left[\alpha \frac{Y_{t+1}}{K_t} + (1-\delta)\right] = 0 \quad (17)$$

$$\frac{\partial \mathscr{L}}{\partial N_t} = \beta^0 \left[u'_N(N_t) + \lambda_t (1-\alpha) \frac{Y_t}{N_t}\right] = 0 \quad (18)$$

$$\frac{\partial \mathscr{L}}{\partial \lambda_t} = \beta^0 \left[A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) K_{t-1} - C_t - K_t\right] = 0 \quad (19)$$

First Order Conditions (FOCs) simplified

- Insert $u'_C(C_t) = \lambda_t$, $u'_C(C_{t+1}) = \lambda_{t+1}$ into the FOC $\frac{\partial L}{\partial K_t}$, and get:

$$u'_C(C_t) = \beta\left[u'_C(C_{t+1})R_{t+1}\right] \quad \text{(Euler equation)} \tag{20}$$

- Bring expectations back. Eq. (Euler equation) with uncertainty:

$$E_t\left[u'_C(C_t)\right] = E_t\left[\beta\left(u'_C(C_{t+1})R_{t+1}\right)\right] \tag{21}$$

- The specific utility function can now be applied:

$$u'_C(C_{t+i}) = \frac{\partial u}{\partial C_{t+i}} = C_{t+i}^{-\sigma} \tag{22}$$

- The Euler equation appears as:

$$C_t^{-\sigma} = \beta E_t\left[C_{t+1}^{-\sigma}R_{t+1}\right] \tag{23}$$

More on FOCs

- Notice that from the FOCs $\frac{\partial L}{\partial C_t} = 0$, $\frac{\partial L}{\partial N_t} = 0$ we can get another result by cancelling out λ_t:

$$\beta^t \left[u'_N(N_t) - \lambda_t (1-\alpha) \frac{Y_t}{N_t} \right] = 0 \quad (24)$$

- As $\beta^t \neq 0$, therefore:

$$u'_N(N_t) - \lambda_t (1-\alpha) \frac{Y_t}{N_t} = 0 \quad (25)$$

- But as $u'_N(N_t) = \theta$, and $\lambda_t = u'_C(C_t)$, we get:

$$\frac{Y_t}{N_t} = \frac{\theta}{1-\alpha} C_t^{\sigma} \quad (26)$$

The maximization of utility: 4 equations x 5 variables

- The FOCs give us 3 equations involving 5 variables (Y_{t+i}, N_{t+i}, C_{t+i}, R_{t+i}, K_{t+i}):

$$\frac{Y_t}{N_t} = \frac{\theta}{1-\alpha} C_t^{\sigma} \tag{27}$$

$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma} R_{t+1} \right] \tag{28}$$

$$R_{t+1} = \alpha \frac{Y_{t+1}}{K_t} + (1-\delta) \tag{29}$$

- The system is indeterminate. Two further equations are needed:
  - The production function
  - The capital accumulation

- The system is indeterminate. Two further equations are needed:
  - The production function
  - The capital accumulation

- But these two bring another two variables into the system (A_t, I_t), which requires two further equations:
  - The national accounting identity
  - The TFP process

- Our seven equations are:

$$R_{t+1} = \alpha \left(\frac{Y_{t+1}}{K_t} \right) + (1-\delta) \tag{30}$$

$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma} R_{t+1} \right] \tag{31}$$

$$\frac{Y_t}{N_t} = \frac{\theta}{1-\alpha} C_t^{\sigma} \tag{32}$$

$$K_t = (1-\delta) K_{t-1} + I_t \tag{33}$$

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} \tag{34}$$

$$C_t + I_t = Y_t \tag{35}$$

$$\ln A_t = (1-\rho) \ln \bar{A} + \rho \ln A_{t-1} + \varepsilon_t \tag{36}$$

- A nonlinear system of stochastic difference equations (some of them are nonlinear)
- Solutions are extremely difficult (if not impossible) to be obtained for these systems
- A trick: linearize the system in the vicinity of the steady state. Widely used and very useful in

Linearization: what is it?

- We shall recall a number of points:
  - The system has 7 endogenous variables (Y_{t+i}, N_{t+i}, C_{t+i}, R_{t+i}, K_{t+i}, A_{t+i}, I_{t+i})
  - In steady state, for any variable v_t, we get: $v_t = v_{t+1} = \bar{v}$
  - The natural way to linearize an equation is to apply logs, or $\Delta \log$ (first difference in logs)
  - Remember that $\Delta \log$ is approximately equal to a growth rate
- We will apply $\Delta \log$ to our system
- Linearization may look very complicated, but in fact it's extremely simple
- We only need to know how to transform the equations of the model into $\Delta \log$ functions

- Transforming functions into log-differences: first case
- A linear function: $Y_t = 2X_t$. Apply logs to two consecutive periods:

$$\ln Y_t = \ln 2 + \ln X_t \tag{37}$$

$$\ln Y_{t+1} = \ln 2 + \ln X_{t+1} \tag{38}$$

- Therefore, the first difference of logs is:

$$\ln Y_{t+1} - \ln Y_t = (\ln 2 + \ln X_{t+1}) - (\ln 2 + \ln X_t) = \ln X_{t+1} - \ln X_t \tag{39}$$

- In this kind of function, the growth rate of Y, let's call it y, is equal to the growth rate of X, x:

$$y = x \tag{40}$$

- Don't forget: we use small letters to express the growth rate of a variable

Transforming functions into log-differences: second case

- A linear function of two independent variables: $Y_t = 2X_t Z_t$. Apply logs to two consecutive periods, and you will get:

$$y = x + z \tag{41}$$

- Prove this result yourself.

Transforming functions into log-differences: third case

- A power function: $Y_t = 2X_t Z_t^3$. Apply logs:

$$\ln Y_t = \ln 2 + \ln X_t + 3 \ln Z_t \tag{42}$$

$$\ln Y_{t+1} = \ln 2 + \ln X_{t+1} + 3 \ln Z_{t+1} \tag{43}$$

- Therefore, the first difference of logs is:

$$\ln Y_{t+1} - \ln Y_t = (\ln 2 + \ln X_{t+1} + 3 \ln Z_{t+1}) - (\ln 2 + \ln X_t + 3 \ln Z_t) = \ln X_{t+1} - \ln X_t + 3(\ln Z_{t+1} - \ln Z_t) \tag{44}$$

- So this power function can be written in $\Delta \log$ as:

$$y = x + 3z \tag{45}$$

Transforming functions into log-differences: fourth case

- The last function we need to consider is an additive function like $Y_{t+1} = X_{t+1} + Z_{t+1}$. Here we can't apply logs. But there is another way:

$$\frac{Y_{t+1}}{Y_t} = \frac{X_{t+1}}{X_t}\frac{X_t}{Y_t} + \frac{Z_{t+1}}{Z_t}\frac{Z_t}{Y_t} \tag{46}$$

- Now apply the following:

$$\frac{Y_{t+1}}{Y_t} = 1 + y, \quad \frac{X_{t+1}}{X_t} = 1 + x, \quad \frac{Z_{t+1}}{Z_t} = 1 + z \tag{47}$$

- The previous equation can be written as:

$$(1+y)Y_t = (1+x)X_t + (1+z)Z_t \tag{48}$$

- Divide through by Y_t and get:

$$1 + y = (1+x)\frac{X_t}{Y_t} + (1+z)\frac{Z_t}{Y_t} \tag{49}$$

Transforming functions into log-differences: fourth case (cont.)

- Notice that the previous equation can be written as:

$$1 + y = \left(\frac{X_t}{Y_t} + \frac{Z_t}{Y_t} \right) \tag{50}$$

- Notice that the previous equation can be written as:

$$1 + y = \left(\frac{X_t}{Y_t} + \frac{Z_t}{Y_t} \right) + x\frac{X_t}{Y_t} + z\frac{Z_t}{Y_t} \tag{51}$$

- Therefore, an additive function like $Y_{t+1} = X_{t+1} + Z_{t+1}$ can be expressed as:

$$y = x\frac{X_t}{Y_t} + z\frac{Z_t}{Y_t} \tag{52}$$

- Notice that if $Z = 2$, its growth rate were $z = 0$, and we would get:

$$y = x\frac{X_t}{Y_t} \tag{53}$$

- Let's summarize our results:

Variables in levels	Variables in Δ log
$Y_t = 2X_t$	$y = x$
$Y_t = 2X_t Z_t$	$y = x + z$
$Y_t = 2X_t Z_t^3$	$y = x + 3z$
$Y_{t+1} = X_{t+1} + Z_{t+1}$	$y = x\frac{X_t}{Y_t} + z\frac{Z_t}{Y_t}$
$Y_{t+1} = X_{t+1} + 2$	$y = x\frac{X_t}{Y_t}$

Linearizing the model in the vicinity of the steady state

- Transforming our system into a linear one:

$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma} R_{t+1} \right], \quad c_t = E_t c_{t+1} - \frac{1}{\sigma} E_t r_{t+1} \tag{54}$$

$$\frac{Y_t}{N_t} = \frac{\theta}{1-\alpha} C_t^{\sigma}, \quad n_t = y_t - \sigma c_t \tag{55}$$

$$K_t = (1-\delta) K_{t-1} + I_t, \quad k_t = (1-\delta) k_{t-1} + i_t \tag{56}$$

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \quad y_t = a_t + \alpha k_{t-1} + (1-\alpha) n_t \tag{57}$$

$$C_t + I_t = Y_t, \quad y_t = c_t \frac{C_t}{Y_t} + i_t \frac{I_t}{Y_t} \tag{58}$$

$$R_t = \alpha \left(\frac{Y_t}{K_{t-1}} \right) + (1-\delta), \quad r_t = \alpha \left(\frac{R_t Y_t}{K_{t-1}} \right) (y_t - k_{t-1}) \tag{59}$$

$$\ln A_t = (1-\rho) \ln \bar{A} + \rho \ln A_{t-1} + \varepsilon_t, \quad a_t = \rho a_{t-1} + \varepsilon_t \tag{60}$$

- Notice that now our system is: 7 equations, 12 unknowns: (c, r, n, y, k, i, a) plus (K, C, Y, I, R)

Linearization: one example

- One example. Let us solve the less simple equation of the whole set:

$$R_t = \alpha\left(\frac{Y_t}{K_{t-1}}\right) + (1-\delta) \tag{61}$$

- Simplify the previous equation by assuming that $Z_t = \frac{Y_t}{K_{t-1}}$, and $\mu = 1-\delta$:

$$R_t = \alpha Z_t + \mu \tag{62}$$

- Now apply the rule discussed above and get:

$$r_t = \alpha z_t \frac{Z_t}{R_t} \tag{63}$$

- But as $z_t = y_t - k_{t-1}$:

$$r_t = \alpha\left(\frac{R_t Y_t}{K_{t-1}}\right)(y_t - k_{t-1}) \tag{64}$$

Determining the steady state

- We can determine the values of K, C, Y, I, R associated with the steady state.
- Remember that in the vicinity of the steady state, for any x_t, we get $x_t = x_{t+1} = \bar{x}$, then $\frac{x_t}{x_{t+1}} = 1$.
- Let's start with the Euler equation, as $C_t = C_{t+1} = \bar{C}$, then:

$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma} R_{t+1} \right] \tag{65}$$

- If $\bar{R} = \beta^{-1}$, then from the production function:

$$\bar{R} = \alpha \left(\frac{\bar{Y}}{\bar{K}} \right) + (1 - \delta) \tag{66}$$

- Therefore:

$$\frac{\bar{Y}}{\bar{K}} = \frac{\beta^{-1} - (1 - \delta)}{\alpha} \tag{67}$$

Determining the steady state (continued)

- As we know that $\bar{R} = \beta^{-1}$ and $\frac{\bar{Y}}{\bar{K}} = \frac{\beta^{-1} - (1-\delta)}{\alpha}$, then:

$$\alpha \bar{R} \frac{\bar{Y}}{\bar{K}} = \frac{1}{\beta(1-\delta)} \tag{68}$$

- Next, from the capital accumulation equation:

$$\bar{K} = (1-\delta)\bar{K} + \bar{I} \tag{69}$$

- Therefore:

$$\frac{\bar{I}}{\bar{K}} = \delta \tag{70}$$

- And:

$$\frac{\bar{I}}{\bar{Y}} = \frac{\bar{I}}{\bar{K}} \frac{\bar{K}}{\bar{Y}} = \phi, \quad \text{for simplicity with} \quad \phi = \frac{\alpha\delta}{\beta^{-1} - (1-\delta)} \tag{71}$$

- Finally:

Summary: our linearized model in the vicinity of the steady state

- Our system of stochastic linear difference equations with rational expectations looks like:

$$c_t = E_t c_{t+1} - \frac{1}{\sigma} E_t r_{t+1} \tag{74}$$

$$n_t = y_t - \sigma c_t \tag{75}$$

$$k_t = (1-\delta)k_{t-1} + \delta i_t \tag{76}$$

$$y_t = a_t + \alpha k_{t-1} + (1-\alpha) n_t \tag{77}$$

$$y_t = c_t(1-\phi) + \phi i_t \tag{78}$$

$$r_t = \left[\frac{1}{\beta(1-\delta)}\right](y_t - k_{t-1}) \tag{79}$$

$$a_t = \rho a_{t-1} + \varepsilon_t \tag{80}$$

- With $\phi = \frac{\alpha\delta}{\beta^{-1}-(1-\delta)}$.

Numerical simulation of the linearized model

- Now we can give numbers to the parameters, take the model to the computer and simulate the impact of shocks upon the endogenous variables.
- We use a routine for Matlab developed by Harald Uhlig, now at the University of Chicago.
- Calibrate the model: $\alpha = 0.4$, $\delta = 0.012$, $\rho = 0.95$, $\beta = 0.987$, $\sigma_e = 0.07$, $\sigma = 1$ and $\bar{N} = 1/3$ (steady state employment is a third of total time endowment).
- See next figures.

Output vs consumption

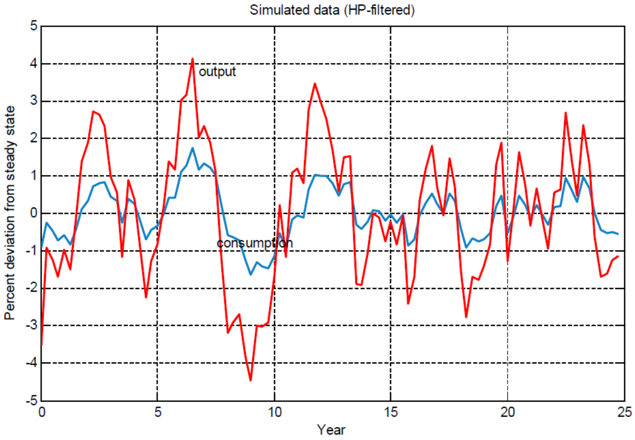

Figure: Simulated data (HP-filtered): Output vs Consumption

Output vs TFP

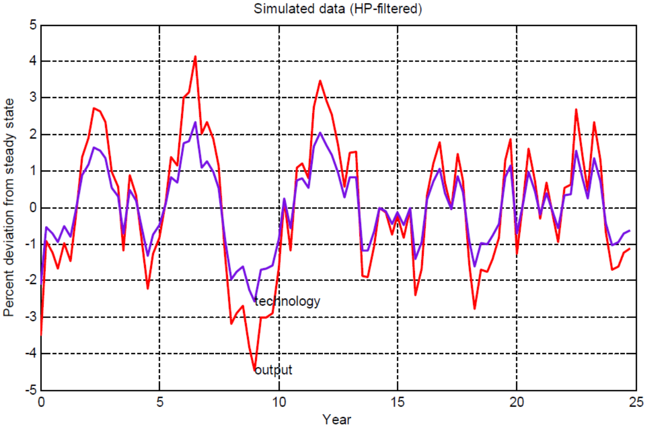

Figure: Simulated data (HP-filtered): Output vs TFP

Capital, interest rate, TFP and labor

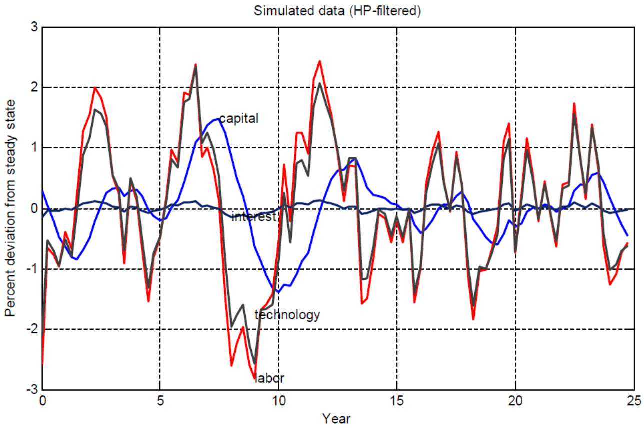

Figure: Simulated data (HP-filtered): Capital, Interest Rate, TFP and Labor

Impulse response functions: A positive technological shock

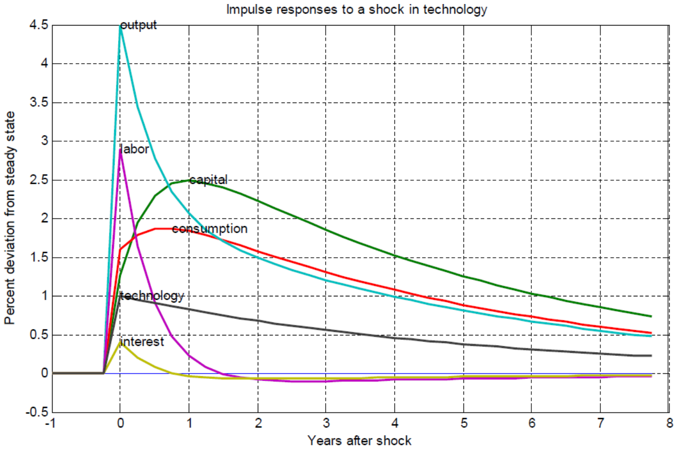

Figure: Impulse responses to a shock in technology

Impulse response functions: A one percent increase in capital

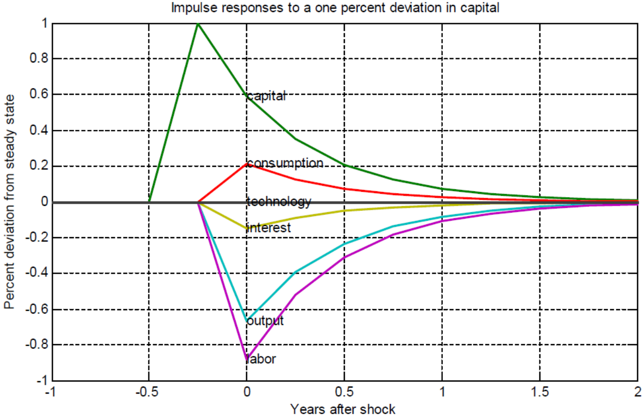

Figure: Impulse responses to a one percent deviation in capital

The RBC model: shortcomings

- Reproduces relatively well several stylized facts of business cycles:
  - Output is nearly as volatile as in the data.
  - Consumption is less volatile than output.
  - Investment is more volatile.
  - Persistence is high.
- It seems OK with covariances.
- Serious problems:
  - Variability of hours of work is understated as well as consumption.
  - Real wages and interest rates are highly procyclical (not so in the data).
  - Where do the negative shocks come from?
  - No role for monetary policy.
  - Fiscal policy is of little help due to Ricardian equivalence.

Readings

- Eric Sims (2017). Graduate Macro Theory II: The Real Business Cycle Model, University of Notre Dame, Spring 2017.
- Dirk Krueger (2007). ”Quantitative Macroeconomics: An Introduction” Unpublished manuscript, Department of Economics University of Pennsylvania.