ECON 710 - Advanced Macroeconomics

Komla Avoumatsodo

Standard Neoclassical Model

Contents

1 Foundations of the Neoclassical Model

In the Solow model, the savings rate is constant and exogenous. In the neoclassical growth model, also known as the Ramsey model or the Cass-Koopmans model, the preferences (utilities) of households are clearly specified. Households choose consumption and investment optimally to maximize their utility. The key difference between the Solow model and the neoclassical growth model is the endogenous treatment of savings and labor supply.

2 Preferences

In the neoclassical growth model, we consider a finite horizon and discrete time. The household derives utility from consumption and leisure. Let $c_t \equiv C_t/L_t$ and z_t be per capita consumption and leisure at time t, where C_t is total consumption and L_t is the population size.

Preferences are defined over consumption and leisure paths, $\mathbf{x} = \{x_t\}_{t=0}^{\infty}$, with $x_t = (c_t, z_t)$, and are represented by the utility function:

$$\begin{aligned} \mathscr{U}: \quad \mathbb{X}^{\infty} &\to \mathbb{R} \\ x &\mapsto \mathscr{U}(x_0, x_1, \dots) \end{aligned}$$

where $\mathbb{X}$ is the domain of x_t and is typically $\mathbb{R}_+ \times [0,1]$.

Preferences are said to be recursive if there exists a function $W : \mathbb{X} \times \mathbb{R} \to \mathbb{R}$ (often called a utility aggregator) such that, for any $\{x_t\}_{t=0}^{\infty}$,

$$\mathscr{U}(x_0, x_1, \dots) = W[x_0, \mathscr{U}(x_1, x_2, \dots)].$$

Thus, each $\{x_t\}_{t=0}^{\infty}$ induces a utility path $\{\mathscr{U}_t\}_{t=0}^{\infty}$ according to the following recursion:

$$\mathscr{U}_t = W(x_t, \mathscr{U}_{t+1}).$$

Preferences are additively separable if there exist functions v_t such that:

$$\mathscr{U}(\mathbf{x}) = \sum_{t=0}^{\infty} v_t(x_t).$$

where $v_t(x_t)$ represents the utility at period 0 from consumption and leisure at period t.

We will assume that preferences are recursive and additively separable. This implies that the utility aggregator W must be linear in its second argument. Therefore, there exists a function $U : \mathbb{R} \to \mathbb{R}$ and a scalar $\beta \in \mathbb{R}$ such that:

$$W(x, y) = U(x) + \beta y.$$

This implies that:

$$\mathscr{U}_t = U(x_t) + \beta \mathscr{U}_{t+1},$$

or equivalently,

$$\mathscr{U}_t = \sum_{\tau=0}^{\infty} \beta^{\tau} U(x_{t+\tau}).$$

Here, $\beta \in (0,1)$ is the discount factor and U is called the instantaneous utility function.

We assume that the maximum amount of time per period is 1. Thus, $\mathbb{X} = \mathbb{R}_+ \times [0,1]$.

In the special case $t = 0$,

$$\mathscr{U}_0 = \sum_{\tau=0}^{\infty} \beta^{\tau} U(x_{\tau}) = \sum_{\tau=0}^{\infty} \beta^{\tau} U(c_{\tau}, z_{\tau}).$$

$\mathscr{U}_0$ is the function that the household seeks to optimize by choosing a path of consumption and leisure $\{c_t, z_t\}_{t=0}^{\infty}$.

Assumption 1. *The function U must be neoclassical, i.e.:*

- *Continuous and twice differentiable.*
- *Strictly increasing and strictly concave:*

$$U_c(c,z) > 0 > U_{cc}(c,z),$$
$$U_z(c,z) > 0 > U_{zz}(c,z),$$
$$U_{cz}^2 < U_{cc} U_{zz}.$$

- *Satisfies the Inada conditions:*

$$\lim_{c \to 0} U_c = \infty, \quad \lim_{c \to \infty} U_c = 0, \quad \lim_{z \to 0} U_z = \infty, \quad \textit{and} \quad \lim_{z \to 1} U_z = 0.$$

3 Production Technology

All firms have access to the same production technology, which implies that the economy has a representative firm. Factor and product markets are competitive. The aggregate production function for the single final good is:

$$Y_t = F(K_t, L_t, A_t) \tag{1}$$

where K_t and L_t represent the demand for capital and labor at time t, and A_t is the technology at time t.

Assumption 2 (Continuity, Differentiability, Diminishing and Positive Marginal Products, and Constant Returns to Scale)**.** *The production function $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ is twice differentiable in K_t*

and L_t, and satisfies:

$$F_K(K_t, L_t, A_t) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K_t, L_t, A_t) \equiv \frac{\partial F(\cdot)}{\partial L} > 0,$$

$$F_{KK}(K_t, L_t, A_t) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K_t, L_t, A_t) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0.$$

Furthermore, F has constant returns to scale in K_t and L_t.

Assumption 3 (Inada Conditions)**.** *The production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfies the Inada conditions:*

$$\lim_{K\to 0} F_K(K_t, L_t, A_t) = \infty \quad \textit{and} \quad \lim_{K\to\infty} F_K(K_t, L_t, A_t) = 0, \quad \forall L_t > 0,$$

$$\lim_{L\to 0} F_L(K_t, L_t, A_t) = \infty \quad \textit{and} \quad \lim_{L\to\infty} F_L(K_t, L_t, A_t) = 0, \quad \forall K_t > 0.$$

These conditions ensure the existence of interior equilibria and imply that all factors of production are necessary, i.e.,

$$F(0, L_t, A_t) = F(K_t, 0, A_t) = 0.$$

A production function that satisfies Assumptions 2 and 3 is called a **neoclassical production function or technology**. In intensive notation (i.e., the production function per worker):

$$y_t \equiv \frac{Y_t}{L} = F\left(\frac{K_t}{L}, \frac{L_t}{L}, A_t\right) \equiv F(k_t, \ell_t),$$

where A_t is assumed constant (equal to 1), $k_t \equiv K_t/L$, and $\ell_t \equiv L_t/L$.

Factor prices are given by:

$$R_t = F_K(k_t, \ell_t),$$

$$w_t = F_L(k_t, \ell_t).$$

The time constraint is given by:

$$\ell_t + z_t \leq 1,$$

where z_t and ℓ_t are interpreted as the fractions of time the household spends on leisure and work, respectively.

Since the economy is closed, the resource constraint (per worker) is given by:

$$c_t + i_t \leq y_t. \tag{2}$$

where i_t is the investment per worker at time t.

3.1 Technology and Resource Constraint

The law of motion for the capital stock is:

$$k_{t+1} = i_t + (1-\delta)k_t. \tag{3}$$

By combining equations (2) and (3), we obtain:

$$c_t + k_{t+1} \leq F(k_t, \ell_t) + (1-\delta)k_t.$$

Finally, we impose the following natural non-negativity constraints:

$$c_t \geq 0, \quad \ell_t \geq 0, \quad z_t \geq 0, \quad k_t \geq 0.$$

In equilibrium, the time constraint will be saturated, which implies:

$$\ell_t = 1 - z_t.$$

4 Social Planner's Problem

We begin the analysis of the neoclassical growth model by considering the optimal allocation of a benevolent social planner. The social planner (SP) chooses the static and intertemporal allocation of resources in the economy to maximize social welfare. We will later determine the allocations in a decentralized competitive market environment and show that the two allocations coincide.

4.1 Formulation of the Social Planner's Problem

The SP chooses a path $\{c_t, \ell_t, k_{t+1}\}_{t=0}^{\infty}$ that maximizes utility subject to the economy's resource constraint, with a given initial $k_0 > 0$:

$$\begin{aligned} \max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{\infty}} \quad & \mathbb{U}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t) \\ \text{s.t.} \quad & c_t + k_{t+1} \leq (1-\delta)k_t + F(k_t, \ell_t), \quad \forall t \geq 0, \\ & c_t \geq 0, \quad \ell_t \in [0,1], \quad k_{t+1} \geq 0, \quad \forall t \geq 0, \\ & k_0 > 0 \text{ given.} \end{aligned}$$

The household will never choose $c_t = 0$, $k_{t+1} = 0$, $\ell_t = 0$, or $\ell_t = 1$. Therefore, the constraints $c_t \geq 0$, $\ell_t \in [0,1]$, $k_{t+1} \geq 0$ will often be ignored in the solution.

4.2 Lagrangian and First-Order Conditions

Let μ_t denote the Lagrange multiplier for the resource constraint. The Lagrangian for the social planner's problem is written as follows:

$$\mathscr{L}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t) + \sum_{t=0}^{\infty} \mu_t [(1-\delta)k_t + F(k_t, \ell_t) - k_{t+1} - c_t].$$

The first-order conditions (FOCs) are:

$$\frac{\partial \mathscr{L}_0}{\partial c_t} = \beta^t U_c(c_t, 1-\ell_t) - \mu_t = 0,$$

$$\frac{\partial \mathscr{L}_0}{\partial \ell_t} = -\beta^t U_z(c_t, 1-\ell_t) + \mu_t F_L(k_t, \ell_t) = 0,$$

$$\frac{\partial \mathscr{L}_0}{\partial k_{t+1}} = -\mu_t + \mu_{t+1}[(1-\delta) + F_K(k_{t+1}, \ell_{t+1})] = 0.$$

4.3 Characterization of Equilibrium

By combining the above conditions, we obtain:

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = F_L(k_t, \ell_t), \tag{4}$$

$$\frac{U_c(c_t, 1-\ell_t)}{\beta U_c(c_{t+1}, 1-\ell_{t+1})} = 1 - \delta + F_K(k_{t+1}, \ell_{t+1}). \tag{5}$$

Equation (4) means that the marginal rate of substitution between consumption and leisure equals the marginal product of labor. Equation (5) equates the intertemporal marginal rate of substitution in consumption to the net marginal product of capital (including depreciation). The latter condition is called the Euler equation.

4.4 Interpretation of Equilibrium Conditions

Interpretation of equation (4):

$$\underbrace{U_z(c_t, 1-\ell_t)}_{\text{Disutility from working one unit of time}} = \underbrace{F_L(k_t, \ell_t)}_{\text{Income from one unit of work}} \times \underbrace{U_c(c_t, 1-\ell_t)}_{\text{Utility from consuming \$1}}.$$

Interpretation of the Euler equation (5):

$$\underbrace{U_c(c_t, 1-\ell_t)}_{\text{Utility lost by saving \$1}} = \underbrace{[1-\delta + F_K(k_{t+1}, \ell_{t+1})]}_{\text{Return in } t+1 \text{ on \$1 invested in } t} \times \underbrace{\beta U_c(c_{t+1}, 1-\ell_{t+1})}_{\text{Utility of consuming \$1 in } t+1}.$$

- Impatience: If β decreases, current consumption increases, and future consumption decreases.
- Return on investment: If it increases, more saving occurs for greater future consumption.

4.5 Social Planner's Problem: Finite Horizon

Suppose the horizon is finite, $T < \infty$. The social planner's problem can be written as:

$$\begin{aligned}
&\max_{\{c_t,\ell_t,k_{t+1}\}_{t=0}^{T}} \mathbb{U}_0 = \sum_{t=0}^{T} \beta^t U(c_t, 1-\ell_t) \\
&\text{s.t.} \quad c_t + k_{t+1} \leq (1-\delta)k_t + F(k_t,\ell_t), \quad \forall t = 0,\cdots,T, \\
&\qquad\qquad c_t \geq 0, \quad \ell_t \in [0,1], \quad k_{t+1} \geq 0, \quad \forall t = 0,\cdots,T, \\
&\qquad\qquad k_0 > 0 \text{ given.}
\end{aligned}$$

For $k_{t+1} \geq 0$, the decision becomes non-trivial due to the choice at T. We need to introduce a multiplier λ_t for this constraint. The Lagrangian for the social planner's problem is:

$$\mathscr{L}_0 = \sum_{t=0}^{T} \beta^t U(c_t, 1-\ell_t) + \sum_{t=0}^{T} \mu_t[(1-\delta)k_t + F(k_t,\ell_t) - k_{t+1} - c_t] + \sum_{t=0}^{T} \lambda_t k_{t+1}.$$

We calculate the FOCs and solve the problem. We can also use the Kuhn-Tucker conditions for optimization with non-negativity constraints.

4.6 Kuhn-Tucker Theorem

Assume x^* maximizes the following problem:

$$\begin{aligned}
&\max_{x\in\mathbb{R}^n} f(x) \\
&\text{s.t. } g_1(x) = b_1, \ldots, g_M(x) = b_M, \\
&\qquad h_1(x) \leq c_1, \ldots, h_K(x) \leq c_K.
\end{aligned}$$

This is a constrained maximization problem with M equality constraints and K inequality constraints. Assume the constraint qualification condition is satisfied at x^*.

The Lagrangian is then given by:

$$\mathscr{L} = f(x) + \sum_{m=1}^{M} \lambda_m (b_m - g_m(x)) + \sum_{k=1}^{K} \mu_k (c_k - h_k(x)).$$

First-order conditions:

$$\frac{\partial L(x^*)}{\partial x_n} = \frac{\partial f(x^*)}{\partial x_n} - \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x^*)}{\partial x_n} - \sum_{k=1}^{K} \mu_k \frac{\partial h_k(x^*)}{\partial x_n} = 0,$$

for all $n = 1, \ldots, N$, and

$$h_k(x) \leq c_k, \quad \mu_k \geq 0, \quad \text{and} \quad \mu_k(c_k - h_k(x^*)) = 0,$$

for all $k = 1, \ldots, K$.

4.7 Application to the Social Planner's Problem

Return to the Social Planner's (SP) problem. The Kuhn-Tucker conditions with respect to k_{T+1} are written as:

$$\frac{\partial \mathscr{L}_0}{\partial k_{T+1}} \geq 0, \quad k_{T+1} \geq 0, \quad \text{and} \quad \frac{\partial \mathscr{L}_0}{\partial k_{T+1}} k_{T+1} = 0.$$

This implies that $\lambda_T \geq 0$, $k_{T+1} \geq 0$, and $\lambda_T k_{T+1} = 0$.

This implies that $k_{T+1} = 0$, meaning the shadow value of k_{T+1} is zero. When $T = \infty$, the terminal condition $\mu_T k_{T+1} = 0$ is replaced by the **transversality condition**:

$$\lim_{t \to \infty} \mu_t k_{t+1} = 0.$$

This means that the discounted shadow value of capital converges to zero:

$$\lim_{t \to \infty} \beta^t U_c(c_t, \ell_t) k_{t+1} = 0.$$

Proposition 4.1. *The path $\{c_t, \ell_t, k_{t+1}\}_{t=0}^{\infty}$ is a solution to the social planner's problem if and only if the following conditions hold for all $t \geq 0$:*

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = F_L(k_t, \ell_t),$$

$$\frac{U_c(c_t, 1-\ell_t)}{\beta U_c(c_{t+1}, 1-\ell_{t+1})} = 1 - \delta + F_K(k_{t+1}, \ell_{t+1}),$$

$$k_{t+1} = F(k_t, \ell_t) + (1-\delta)k_t - c_t.$$

The initial condition is:

$$k_0 > 0 \quad \textit{(given)}.$$

The transversality condition is:

$$\lim_{t \to \infty} \beta^t U_c(c_t, 1-\ell_t) k_{t+1} = 0.$$

4.8 Application Example

Consider the following utility function:

$$u(c_t, \ell_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \frac{\ell_t^{\gamma}}{1+\gamma}.$$

- u is additively separable in consumption and leisure.
- The intertemporal marginal rate of substitution of consumption is:

$$MRS = \frac{\beta u'(c_{t+1})}{u'(c_t)}.$$

- The elasticity of intertemporal substitution in consumption is:

$$\frac{\partial(c_{t+1}/c_t)}{\partial MRS} \cdot \frac{MRS}{c_{t+1}/c_t} = \frac{1}{\sigma}.$$

Consider the following neoclassical production function:

$$Y_t = F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha},$$

where $0 < \alpha < 1$. F is a Cobb-Douglas production function. Output per worker is given by:

$$y_t = \frac{Y_t}{L_t} = Ak_t^{\alpha} \ell_t^{1-\alpha}.$$

Formulate the Social Planner's problem and solve it to derive the Euler equation in finite and infinite horizon.

5 Competitive Equilibrium

In the optimal equilibrium, the social planner decides on allocations in the economy. This means that a central authority determines how resources are distributed to achieve the best possible outcome for society.

In contrast, a competitive equilibrium is a state where prices and quantities are determined by the interactions of households and firms in the market. In this scenario:

- **Households**: They choose quantities of goods and services that maximize their utility (satisfaction) given their budget constraints. They take prices as given and cannot influence them.
- **Firms**: They decide on the level of production and the quantities of inputs (like labor and capital) that maximize their profit. They also take prices as given.

- **Markets**: Prices adjust so that supply equals demand in all markets, ensuring that there is no excess supply or demand.

5.1 Household Preferences

We consider a representative household, meaning we analyze the behavior of a typical household rather than individual households. We assume there is no population growth, simplifying the analysis.

The household's preferences are represented by a utility function, which measures the satisfaction or happiness derived from consuming goods and services. The utility function is given by:

$$\mathbb{U}_0 = \sum_{t=0}^{\infty} \beta^t U\left(c_t, z_t\right) \tag{6}$$

Here, $\mathbb{U}_0$ is the total utility, β is the discount factor (reflecting the household's preference for current consumption over future consumption), c_t is consumption at time t, and z_t is leisure at time t.

The household's time constraint is given by:

$$z_t = 1 - \ell_t \tag{7}$$

where ℓ_t is the labor supplied by the household at time t.

5.2 Household Budget Constraint

The household's budget constraint represents the trade-off between consumption, investment, and income. It is given by:

$$c_t + i_t + x_t \leq r_t k_t + R_t b_t + w_t \ell_t + \alpha \Pi_t \tag{8}$$

Here:

- r_t is the rental rate of capital.
- w_t is the wage rate.
- R_t is the interest rate on risk-free bonds.
- α is the share of profit Π_t paid to the household.
- x_t is the investment in bonds.

The household accumulates capital according to the law of motion:

$$k_{t+1} = (1-\delta)k_t + i_t \tag{9}$$

and bonds according to:

$$b_{t+1} = b_t + x_t \tag{10}$$

In equilibrium, firm profits are zero due to perfect competition, so $\Pi_t = 0$. The budget constraint can be rewritten as:

$$c_t + k_{t+1} + b_{t+1} \leq (1-\delta + r_t)\, k_t + (1+R_t)\, b_t + w_t \ell_t \tag{11}$$

5.3 Household Debt Limit : No Ponzi Game Condtion

The household faces a non-negativity constraint on capital:

$$k_{t+1} \geq 0 \tag{12}$$

There is no sign constraint on bonds, meaning the household can either lend or borrow. The natural borrowing constraint, or "No Ponzi Game" condition, ensures that the household's net debt does not exceed the present value of its future income:

$$-(1+R_{t+1})\, b_{t+1} \leq (1-\delta + r_{t+1})\, k_{t+1} + \sum_{\tau=t+1}^{\infty} \frac{q_\tau}{q_{t+1}} w_\tau \tag{13}$$

where:

$$q_t \equiv \frac{1}{(1+R_0)(1+R_1)\dots(1+R_t)} = (1+R_{t+1})\, q_{t+1} \tag{14}$$

The arbitrage condition between bonds and capital implies that in equilibrium:

$$R_t = r_t - \delta \tag{15}$$

If $R_t < r_t - \delta$, there would be an excess supply of bonds. If $R_t > r_t - \delta$, no one would invest in capital. Therefore, the household is indifferent between bonds and capital.

If we consider total assets $a_t = b_t + k_t$, the budget constraint simplifies to:

$$c_t + a_{t+1} \leq (1+R_t)\, a_t + w_t \ell_t \tag{16}$$

The natural borrowing constraint becomes:

$$a_{t+1} \geq \underline{a}_{t+1} \tag{17}$$

where:

$$\underline{a}_{t+1} \equiv -\frac{1}{q_t}\sum_{\tau=t+1}^{\infty} q_\tau w_\tau = -\sum_{\tau=t+1}^{\infty}\left[\prod_{j=t+1}^{\tau}\frac{1}{1+R_j}\right]w_\tau \tag{18}$$

We assume $\underline{a}_t$ is bounded, meaning prices $\{R_t, w_t\}_{t=0}^{\infty}$ are such that:

$$\frac{1}{q_t}\sum_{\tau=t+1}^{\infty} q_\tau w_\tau < \infty \tag{19}$$

5.4 Household Problem

Given a sequence of prices $\{R_t, w_t\}_{t=0}^{\infty}$, the household chooses a sequence of $\{c_t, \ell_t, a_{t+1}\}_{t=0}^{\infty}$ to maximize lifetime utility subject to its budget constraints.

$$\begin{aligned} &\max_{\{c_t,\ell_t,a_{t+1}\}_{t=0}^{\infty}} \quad \mathbb{U}_0 = \sum_{t=0}^{\infty}\beta^t U(c_t, 1-\ell_t) \\ &\text{s.t.} \quad c_t + a_{t+1} \leq (1+R_t)\,a_t + w_t\ell_t, \quad \forall t \\ &c_t \geq 0, \quad \ell_t \in [0,1], \quad a_{t+1} \geq \underline{a}_{t+1}, \quad \forall t \end{aligned} \tag{20}$$

If $\mu_t = \beta^t\lambda_t$ is the Lagrange multiplier for the budget constraint, we can write the Lagrangian as follows:

$$\mathscr{L}_0 = \sum_{t=0}^{\infty}\beta^t\left\{U(c_t, 1-\ell_t) + \lambda_t\left[(1+R_t)\,a_t + w_t\ell_t - a_{t+1} - c_t\right]\right\} \tag{21}$$

The first-order condition (FOC) with respect to c_t is:

$$\frac{\partial \mathscr{L}_0}{\partial c_t} = 0 \quad \Leftrightarrow \quad U_c(c_t, z_t) = \lambda_t \tag{22}$$

The FOC with respect to ℓ_t is:

$$\frac{\partial \mathscr{L}_0}{\partial \ell_t} = 0 \quad \Leftrightarrow \quad U_z(c_t, z_t) = \lambda_t w_t \tag{23}$$

These first two FOCs imply that:

$$\frac{U_z(c_t, z_t)}{U_c(c_t, z_t)} = w_t \tag{24}$$

Households equate their marginal rate of substitution between consumption and leisure with the (common) wage rate.

The Kuhn-Tucker conditions with respect to a_{t+1} are written as:

$$\frac{\partial \mathscr{L}_0}{\partial a_{t+1}} = \beta^t\left[-\lambda_t + \beta\,(1+R_{t+1})\,\lambda_{t+1}\right] \leq 0 \tag{25}$$

and

$$a_{t+1} \geq \underline{a}_{t+1}, \quad [\lambda_t - \beta(1+R_{t+1})\lambda_{t+1}]\left[a_{t+1} - \underline{a}_{t+1}\right] = 0 \tag{26}$$

Using $\lambda_t = U_c(c_t, z_t)$, the Euler condition becomes:

$$U_c(c_t, z_t) \geq \beta(1+R_{t+1})U_c(c_{t+1}, z_{t+1}) \tag{27}$$

If $a_{t+1} > \underline{a}_{t+1}$ then $U_c(c_t, z_t) = \beta(1+R_{t+1})U_c(c_{t+1}, z_{t+1})$.

When the borrowing constraint is not binding, households equate their intertemporal marginal rate of substitution with the (common) return on capital.

If $a_{t+1} = \underline{a}_{t+1}$ then $U_c(c_t, z_t) > \beta(1+R_{t+1})U_c(c_{t+1}, z_{t+1})$.

In this case, if $c_\tau = z_\tau = 0$ for all $\tau \geq t$, then we have:

$$U_c(c_{t+1}, z_{t+1}) = \infty \Rightarrow U_c(c_t, z_t) > \infty \tag{28}$$

Which is absurd given that $U_c(c_t, z_t) < \infty$. This shows that the borrowing constraint can never be binding.

The finite horizon Lagrangian is written as:

$$\mathscr{L}_0 = \sum_{t=0}^{T} \beta^t \{U(c_t, 1-\ell_t) + \lambda_t[(1+R_t)a_t + w_t\ell_t - a_{t+1} - c_t]\} \tag{29}$$

The FOC with respect to a_{t+1} for $t = 0, 1 \cdots T-1$ is:

$$-\lambda_t + \beta(1+R_{t+1})\lambda_{t+1} \leq 0 \tag{30}$$

The Kuhn-Tucker condition with respect to a_{T+1} is written as:

$$\lambda_T \geq 0, \qquad a_{T+1} \geq \underline{a}_{T+1}, \quad \lambda_T\left[a_{T+1} - \underline{a}_{T+1}\right] = 0 \tag{31}$$

By multiplying by β^T and letting T tend to infinity, we obtain:

$$\lim_{T\to\infty} \beta^T \lambda_T\left[a_{T+1} - \underline{a}_{T+1}\right] = 0 \tag{32}$$

If the borrowing constraint is never binding, the equation is written as:

$$\lambda_t = \beta[1+R_{t+1}]\lambda_{t+1} \tag{33}$$

This implies that:

$$\beta^t \lambda_t = \prod_{\tau=1}^{t} \frac{1}{1+R_\tau}\lambda_0 = (1+R_0)q_t\lambda_0 \tag{34}$$

We can then rewrite the terminal condition as follows:

$$\lim_{t\to\infty}\beta^t\lambda_t a_{t+1}=\lim_{t\to\infty}\beta^t\lambda_t\underline{a}_{t+1}=(1+R_0)\lambda_0\lim_{t\to\infty}q_t\underline{a}_{t+1} \tag{35}$$

But note that:

$$q_t\underline{a}_{t+1}=\sum_{\tau=t}^{\infty}q_\tau w_\tau \tag{36}$$

Recall that:

$$q_t\underline{a}_{t+1}=\sum_{\tau=t}^{\infty}q_\tau w_\tau, \quad \textit{and} \quad \sum_{\tau=0}^{\infty}q_\tau w_\tau<\infty \tag{37}$$

Then:

$$\lim_{t\to\infty}\sum_{\tau=t}^{\infty}q_\tau w_\tau=0 \tag{38}$$

We obtain the more familiar version of the transversality condition:

$$\lim_{t\to\infty}\beta^t\lambda_t a_{t+1}=0 \tag{39}$$

Which is equivalently written as:

$$\lim_{t\to\infty}\beta^t U_c\left(c_t,1-\ell_t\right)a_{t+1}=0 \tag{40}$$

This allows us to reformulate the household problem in the manner of **Arrow-Debreu**:

$$\begin{aligned}&\max_{\{c_t,\ell_t,k_{t+1},b_{t+1}\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t U\left(c_t,1-\ell_t\right)\\ &\text{s.t. }\sum_{t=0}^{\infty}q_t c_t\le a_0+\sum_{t=0}^{\infty}q_t w_t\ell_t\end{aligned} \tag{41}$$

with

$$a_0+\sum_{t=0}^{\infty}q_t w_t<\infty \tag{42}$$

The intertemporal budget constraint is equivalent to the sequence of period-by-period budget constraints and the natural borrowing limit.

Let $\mu>0$ be the Lagrange multiplier associated with the intertemporal budget. The FOCs with respect to c_t and ℓ_t give:

$$\beta^t U_c\left(c_t,1-\ell_t\right)=\mu q_t \tag{43}$$

and

$$\beta^t U_z(c_t, 1-\ell_t) = \mu q_t w_t \tag{44}$$

We can verify that these conditions coincide with those derived previously.

5.5 Definition

Suppose the sequence of prices $\{R_t, r_t, w_t\}_{t=0}^{\infty}$ satisfies $R_t = r_t - \delta$ for all t, $\sum_{t=0}^{\infty} q_t < \infty$ and $\sum_{t=0}^{\infty} q_t w_t < \infty$.

The sequence $\{c_t, \ell_t, a_t\}_{t=0}^{\infty}$ solves the individual household problem if and only if:

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = w_t, \quad \frac{U_c(c_t, 1-\ell_t)}{\beta U_c(c_{t+1}, 1-\ell_{t+1})} = 1+R_t, \; c_t + a_{t+1} = (1+R_t)a_t + w_t\ell_t, \quad \forall t \tag{45}$$

with $a_0 > 0$ given and:

$$\lim_{t\to\infty} \beta^t U_c(c_t, 1-\ell_t)\, a_{t+1} = 0 \tag{46}$$

Given $\{a_t\}_{t=1}^{\infty}$, an optimal portfolio is any $\{k_t, b_t\}_{t=1}^{\infty}$ such that $k_t \geq 0$ and $b_t = a_t - k_t$. Recall that leisure $z_t = 1 - \ell_t$.

5.6 Firms

We assume there is a representative firm. The representative firm employs labor and rents capital in labor and capital markets. The firm has access to the same technology and produces a homogeneous good that it sells competitively to households.

Let K_t and L_t be the quantities of capital and labor that the firm employs at time t. The firm seeks to maximize its profit at time t:

$$\max_{\{K_t, L_t\}} \Pi_t = F(K_t, L_t) - r_t K_t - w_t L_t \tag{47}$$

The first-order conditions are written as:

$$F_K(K_t, L_t) = r_t, \quad \textit{and} \quad F_L(K_t, L_t) = w_t \tag{48}$$

They imply the capital-labor ratio of each firm (K_t/L_t), but not the size of the firm (L_t). An interior solution to the firms' problem exists if and only if r_t and w_t imply the same K_t/L_t. Since all firms have access to the same technology, they use exactly the same capital-labor ratio. Given that the function F has constant returns to scale, profit is zero in equilibrium:

$$\Pi_t = 0 \tag{49}$$

5.7 Competitive Equilibrium

The bond market is in equilibrium at date t if and only if:

$$0 = b_t \tag{50}$$

with $b_t = a_t - k_t$.

The capital market is in equilibrium at date t if and only if:

$$K_t = k_t \tag{51}$$

The labor market is in equilibrium at date t if and only if:

$$L_t = \ell_t \tag{52}$$

5.8 Definition of Competitive Equilibrium: Arrow-Debreu Formulation

A (competitive) equilibrium of the economy is a sequence of allocations $\{c_t, \ell_t, k_{t+1}, b_{t+1}, K_t, L_t\}_{t=0}^{\infty}$ and prices $\{p_t, R_t, r_t, w_t\}_{t=0}^{\infty}$ such that:

(i) Given $\{p_t, R_t, r_t, w_t\}_{t=0}^{\infty}$, the path $\{c_t, \ell_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}$ solves the household problem:

$$\begin{aligned} \max_{\{c_t, \ell_t, a_{t+1}\}_{t=0}^{\infty}} \quad & \mathbb{U}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t) \\ \text{s.t.} \quad \sum_{t=0}^{\infty} p_t [c_t + a_{t+1}] \leq \sum_{t=0}^{\infty} & p_t [(1+R_t) a_t + w_t \ell_t] \quad \text{with } a_{t+1} \geq \underline{a}_{t+1} \; \forall t \end{aligned} \tag{53}$$

(ii) Given (r_t, w_t), the pair (K_t, L_t) maximizes the firm's profit for each t:

$$\max_{\{K_t, L_t\}} \Pi_t = F(K_t, L_t) - r_t K_t - w_t L_t \tag{54}$$

(iii) The bond, capital, and labor markets are in equilibrium at each period, i.e., equations are satisfied for each date.

5.9 Definition of Competitive Equilibrium: Sequential Formulation

A competitive equilibrium of the economy is a sequence of prices $\{p_t, R_t, r_t, w_t\}_{t=0}^{\infty}$ and allocations $\left\{(c_t, \ell_t, k_{t+1}, b_{t+1})_{j\in[0,L_t]}, (K_t, L_t)\right\}_{t=0}^{\infty}$ such that:

(i) Given $\{R_t, r_t, w_t\}_{t=0}^{\infty}$, the path $\{c_t, \ell_t, k_{t+1}, b_{t+1}\}_{t=0}^{\infty}$ solves the household problem:

$$\begin{aligned} \max_{\{c_t,\ell_t,a_{t+1}\}_{t=0}^{\infty}} \quad & \mathbb{U}_0 = \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t) \\ \text{s.t.} \;\; c_t + a_{t+1} \leq (1+R_t)\, a_t + w_t \ell_t \quad & \forall t \quad \text{with} \;\; \lim_{T\to\infty} q_t a_{t+1} = 0. \end{aligned} \tag{55}$$

(ii) Given (r_t, w_t), the pair (K_t, L_t) maximizes the firm's profit for each t:

$$\max_{\{K_t,L_t\}} \Pi_t = F(K_t, L_t) - r_t K_t - w_t L_t \tag{56}$$

(iii) The bond, capital, and labor markets are in equilibrium at each period, i.e., equations (50), (51), and (52) are satisfied for each date.

5.10 Optimal Equilibrium Implies Competitive Equilibrium

Proposition 5.1. *The set of competitive equilibrium allocations for the market economy coincides with the set of optimal allocations of the social planner.*

The social planner's optimal plan is a sequence of allocations $\{c_t, \ell_t, k_t\}_{t=0}^{\infty}$ such that:

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = F_L(k_t, \ell_t), \quad \forall t \geq 0, \tag{57}$$

$$\frac{U_c(c_t, 1-\ell_t)}{\beta U_c(c_{t+1}, 1-\ell_{t+1})} = [1 - \delta + F_K(k_{t+1}, \ell_{t+1})], \quad \forall t \geq 0 \tag{58}$$

$$c_t + k_{t+1} = (1-\delta)k_t + F(k_t, \ell_t), \quad \forall t \geq 0 \tag{59}$$

$$k_0 > 0 \text{ given, and } \lim_{t\to\infty} \beta^t U_c(c_t, 1-\ell_t)\, k_{t+1} = 0. \tag{60}$$

Let the price path $\{R_t, r_t, w_t\}_{t=0}^{\infty}$ be given by:

$$r_t = F_K(k_t, \ell_t), \tag{61}$$

$$R_t = r_t - \delta, \tag{62}$$

$$w_t = F_L(k_t, \ell_t). \tag{63}$$

Equations (58), (61), and (62) imply:

$$\frac{U_c(c_t, 1-\ell_t)}{\beta U_c(c_{t+1}, 1-\ell_{t+1})} = 1 + R_t. \tag{64}$$

The equilibrium is then given by an allocation $\{c_t, \ell_t, k_t\}_{t=0}^{\infty}$ such that, for all $t \geq 0$:

$$\frac{U_z(c_t, 1-\ell_t)}{U_c(c_t, 1-\ell_t)} = F_L(k_t, \ell_t), \tag{65}$$

$$\frac{U_c(c_t, 1-\ell_t)}{U_c(c_{t+1}, 1-\ell_{t+1})} = \beta\left[1-\delta+F_K(k_{t+1}, \ell_{t+1})\right], \tag{66}$$

$$k_{t+1} = F(k_t, \ell_t) + (1-\delta)k_t - c_t, \tag{67}$$

with $k_0 > 0$ and:

$$\lim_{t\to\infty} \beta^t U_c(c_t, 1-\ell_t)\, k_{t+1} = 0. \tag{68}$$

Finally, the equilibrium prices are given by:

$$R_t = F_K(k_t, \ell_t) - \delta, \quad r_t \equiv F_K(k_t, \ell_t), \quad w_t = F_L(k_t, \ell_t). \tag{69}$$

6 Recursive Competitive Equilibrium

Recursivity : intertemporal maximization is divided into decisions affecting the present and the future (through state variables).

Instead of sequences, a recursive competitive equilibrium is a set of functions of:

- quantities
- values
- prices

These functions describe the agents' choices and prices for given initial conditions.

6.1 Social Planner's Problem

Consider again the social planner's problem. For any $k_0 > 0$, define:

$$V(k_0) \equiv \max_{\{c_t, \ell_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(c_t, 1-\ell_t) \tag{70}$$

subject to the constraints:

$$\begin{gathered} c_t + k_{t+1} \leq (1-\delta)k_t + F(k_t, \ell_t), \quad \forall t \geq 0, \\ c_t, \ell_t, k_{t+1} \geq 0, \quad \forall t \geq 0, \\ k_0 > 0 \text{ given.} \end{gathered} \tag{71}$$

V is called the value function.

The constraint being saturated at equilibrium, we can write:

$$c_t = (1-\delta)k_t + F(k_t, \ell_t) - k_{t+1}, \quad \forall t \geq 0 \tag{72}$$

The value function is then written as:

$$V(k_0) \equiv \max_{\{\ell_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t H(\ell_t, k_t, k_{t+1}) \tag{73}$$

with:

$$H(\ell_t, k_t, k_{t+1}) \equiv U\left[(1-\delta)k_t + F(k_t, \ell_t) - k_{t+1}, 1-\ell_t\right] \tag{74}$$

6.2 Bellman Equation

$$V(k_0) = \max_{\{\ell_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t H(\ell_t, k_t, k_{t+1}) \tag{75}$$

$$V(k_0) = \max_{\ell_0, k_1} \{U(c_0, 1-\ell_0) + \beta V(k_1)\} \tag{76}$$

So we can write in general:

$$V(k_t) = \max \{U(c_t, 1-\ell_t) + \beta V(k_{t+1})\} \tag{77}$$

The Bellman equation for the previous planner's problem is written as:

$$\begin{aligned} V(k) &= \max \left\{U(c, 1-\ell) + \beta V\left(k'\right)\right\} \\ \text{s.t.} \quad & c + k' \leq (1-\delta)k + F(k, \ell) \\ & k' \geq 0, \quad c \in [0, F(k, \ell)], \quad \ell \in [0, 1]. \end{aligned} \tag{78}$$

This is a formulation of the problem in recursive form.

Let $c(k)$, $\ell(k)$, and $k'(k)$ be the values of c, ℓ, and k' that maximize $V(k)$. These expressions are also called policy functions.

6.3 Decentralized Problem

Let's return to the decentralized problem. We use the budget constraint instead of a resource constraint.

Prices are given by the sequential formulation: $\{R_t, w_t\}_{t=0}^{\infty}$ such that:

$$\begin{aligned} R &= R(\bar{K}) \\ w &= w(\bar{K}) \end{aligned} \tag{79}$$

where $\bar{K}$ is the aggregate capital.

Budget constraint in the recursive problem:

$$c + K' = R(\bar{K})K + w(\bar{K})\ell \tag{80}$$

Two variables (states) determine the consumer's choice:

- Their capital K
- The aggregate capital $\bar{K}$, which determines prices

6.4 Household Problem in Recursive Form

The consumer must therefore predict the evolution of aggregate capital. This prediction must be rational: it corresponds to the true law of motion:

$$\bar{K}' = G(\bar{K}) \tag{81}$$

where G is the result of the economy's capital accumulation choices (i.e., the representative consumer in this case).

The household problem in recursive form (Bellman Equation) is then written as:

$$\begin{aligned} V(K,\bar{K}) &= \max_{c,\ell,K'\geq 0} \left\{u(c,1-\ell) + \beta V\left(K',\bar{K}'\right)\right\} \\ \text{s.t. } c + K' &= R(\bar{K})K + w(\bar{K})\ell \\ \bar{K}' &= G(\bar{K}) \end{aligned} \tag{82}$$

6.5 Definition: Recursive Competitive Equilibrium

A recursive competitive equilibrium is a set of functions:

- quantities: $G(\bar{K})$ and $g(K,\bar{K})$
- value: $V(K,\bar{K})$
- prices: $R(\bar{K})$ and $w(\bar{K})$

such that:

- $V(K,\bar{K})$ solves the Bellman equation and $g(K,\bar{K})$ is the associated decision function
- prices are determined competitively:

$$\begin{aligned} R(\bar{K}) &= F_K(\bar{K},L) + 1 - \delta \\ w(\bar{K}) &= F_L(\bar{K},L) \end{aligned} \tag{83}$$

- "consistency":

$$g(\bar{K},\bar{K}) = G(\bar{K}) \tag{84}$$

6.6 Consistency Condition

The consistency condition $G(\bar{K}) = g(\bar{K},\bar{K})$ means that the law of motion perceived by the agent is correct.

In an economy with a single agent, $K = \bar{K}$ implies $G(\bar{K}) = g(\bar{K},\bar{K})$.

With L_t agents:

$$\bar{K} = \sum_{i=1}^{L_t} K_i \tag{85}$$

and

$$G(\bar{K}) = \sum_{i=1}^{N} g_i\,(K_i,\bar{K}) \tag{86}$$

Are the markets in equilibrium? In other words, is the following identity respected?

$$c + K' = F(\bar{K},1) + (1-\delta)\bar{K} \tag{87}$$

6.7 Recursive Equilibrium Condition

The definition of equilibrium tells us that the budget constraint is respected:

$$c + K' = R(\bar{K})K + w(\bar{K})\ell \tag{88}$$

Since all firm revenues go to the consumer, we have:

$$\begin{aligned} c + K' &= F_K(\bar{K},1)K + (1-\delta)K + F_n(\bar{K},1) \\ &= F(\bar{K},1) + (1-\delta)K \end{aligned} \tag{89}$$

(Euler's theorem.) Finally, $K = \bar{K}$ and $g(\bar{K},\bar{K}) = G(\bar{K})$ imply:

$$c + \bar{K}' = F(\bar{K},1) + (1-\delta)\bar{K}. \tag{90}$$

6.8 Characterization: Recursive Competitive Equilibrium

Let's show that the solution to the competitive equilibrium problem is equivalent to those of the previous equilibria.

The Lagrangian for the social planner's problem written in recursive form is:

$$\mathscr{L} = U(c, 1-\ell) + \beta V\left(k'\right) + \lambda\left[(1-\delta)k + F(k,\ell) - k' - c\right] \tag{91}$$

The first-order conditions with respect to c, ℓ, and k' are:

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial c} &= 0 \Leftrightarrow U_c(c,z) = \lambda \\
\frac{\partial \mathscr{L}}{\partial \ell} &= 0 \Leftrightarrow U_z(c,z) = \lambda F_L(k,\ell) \\
\frac{\partial \mathscr{L}}{\partial k'} &= 0 \Leftrightarrow \lambda = \beta V_k\left(k'\right)
\end{aligned} \tag{92}$$

The envelope condition is:

$$V_k(k) = \frac{\partial \mathscr{L}}{\partial k} = \lambda\left[1-\delta + F_K(k,\ell)\right] \tag{93}$$

By combining the two, we conclude:

$$\frac{U_z\left(c_t, 1-\ell_t\right)}{U_c\left(c_t, 1-\ell_t\right)} = F_L\left(k_t, \ell_t\right) \tag{94}$$

and

$$\frac{U_c\left(c_t, \ell_t\right)}{U_c\left(c_{t+1}, \ell_{t+1}\right)} = \beta\left[1-\delta + F_K\left(k_{t+1}, \ell_{t+1}\right)\right] \tag{95}$$