ECON 710 - Advanced Macroeconomics

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Solow Growth Model

Contents

Motivations and Key Facts

To understand the dramatic changes in economic and social conditions over time, let's consider some historical facts about a country at the beginning of the 20th century:

- Life expectancy of a woman was 50 years.
- The infant mortality rate was 18 per 1000 live births.
- 63% of the inhabitants lived outside cities.
- There was no access to running water and electricity for most people.
- 55% of the population was under 20 years old.

These facts describe Canada at the beginning of the 20th century! This highlights the significant progress made in terms of health, urbanization, and overall living standards over the past century. For a historical perspective on the evolution of countries over time, watch this video.

In Search of a Growth Model: The Solow Model

The previous video and the introduction chapter raise two fundamental questions:

☞ How can a country initiate a growth process that will lead it to a higher level of GDP per capita?

☞ Why are some countries rich and others poor?

This chapter presents a model, the **Solow Model** (1956, QJE), that provides some initial answers. We will show how the long-term evolution of income and consumption per worker in a country is affected by:

- The savings rate and investment.
- Technological progress and the population growth rate.

1 Economic Environment

The Solow model is a dynamic general equilibrium model.

- The economy is assumed to be closed and decentralized.
- The agents in the model are households and firms.
- There is a final good, which is produced with two factors of production: capital and labor.
- Time is discrete, $t \in \{0, 1, 2, \ldots\}$.

2 The Household

The representative household:

- Demands the final good.
- Owns the factors of production (i.e., capital and labor).
- Saves a constant fraction s of disposable income.

3 The Firm: Technology and Production

All firms have access to the same production technology, which implies that the economy admits a representative firm. The aggregate production function for the unique final good is given by:

$$Y_t = F(K_t, L_t, A_t) \tag{1}$$

where K_t and L_t represent the demand for capital and labor at time t, and A_t is the technology at time t. Technology is free, publicly available as a non-exclusive and non-rival good.

Assumption 1. *We assume that $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ is continuous, twice differentiable in K and L, and satisfies:*

$$F_K(K_t, L_t, A_t) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K_t, L_t, A_t) \equiv \frac{\partial F(\cdot)}{\partial L} > 0$$
$$F_{KK}(K_t, L_t, A_t) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K_t, L_t, A_t) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0$$

with constant returns to scale in K_t and L_t, i.e., homogeneous of degree 1.

The marginal products of capital and labor increase with the other factor, i.e., capital and labor are complementary:

$$F_{KL}(K_t, L_t, A_t) = F_{LK}(K_t, L_t, A_t) \equiv \frac{\partial^2 F}{\partial K \partial L}(K_t, L_t, A_t) > 0$$

with $F_{KL}(K_t, L_t, A_t) = F_{LK}(K_t, L_t, A_t)$ by Schwarz's theorem.

Definition 3.1 (Homogeneity). *Let m be an integer. The function $F : \mathbb{R}^{p+2} \to \mathbb{R}$ is homogeneous of degree m in $(K_t, L) \in \mathbb{R}^2$ if and only if :*

$$F(\lambda K_t, \lambda L_t, A_t) = \lambda^m F(K_t, L_t, A_t), \quad \forall \lambda \in \mathbb{R}_+ \text{ and } A \in \mathbb{R}^p.$$

3.1 Euler's Theorem

Euler's Theorem is a fundamental result in the theory of homogeneous functions, which is crucial for understanding the properties of production functions in economic models.

Theorem 1 (Euler's Theorem). *Let $F : \mathbb{R}^{p+2} \to \mathbb{R}$ be a differentiable function with partial derivatives F_K and F_L, and homogeneous of degree m in K and L. Then:*

- $mF(K_t, L_t, A_t) = F_K(K_t, L_t, A_t) \times K_t + F_L(K_t, L_t, A_t) \times L_t \quad \forall K_t, L_t \in \mathbb{R},\ A_t \in \mathbb{R}.$
- *$F_K(K_t, L_t, A_t)$ and $F_L(K_t, L_t, A_t)$ are homogeneous of degree $m-1$ in (K_t, L).*

Proof. Since F is homogeneous of degree m in (K_t, L), for all $\lambda \in \mathbb{R}_+$, we have:

$$F(\lambda K_t, \lambda L_t, A_t) = \lambda^m F(K_t, L_t, A_t).$$

Since F is differentiable, differentiating this equation with respect to λ gives:

$$m\lambda^{m-1} F(K_t, L_t, A_t) = F_K(\lambda K_t, \lambda L_t, A_t) K_t + F_L(\lambda K_t, \lambda L_t, A_t) L_t.$$

Substituting $\lambda = 1$, we obtain the first relation:

$$mF(K_t, L_t, A_t) = F_K(K_t, L_t, A_t) \times K_t + F_L(K_t, L_t, A_t) \times L_t.$$

Differentiating the original equation with respect to K and L and dividing both relations by $\lambda > 0$, we get:

$$F_K(\lambda K_t, \lambda L_t, A_t) = \lambda^{m-1} F_K(K_t, L_t, A_t),$$

$$F_L(\lambda K_t, \lambda L_t, A_t) = \lambda^{m-1} F_L(K_t, L_t, A_t).$$

This shows that $F_K(K_t, L_t, A_t)$ and $F_L(K_t, L_t, A_t)$ are homogeneous of degree $m-1$ in (K_t, L). □

3.2 Firm: Technology and Production (Continued)

The function $F : \mathbb{R}_+^3 \to \mathbb{R}_+$ must satisfy the Inada conditions:

$$\lim_{K_t \to 0} F_K(K_t, L_t, A_t) = \infty \quad \text{and} \quad \lim_{K_t \to \infty} F_K(K_t, L_t, A_t) = 0, \quad \forall L_t > 0,$$

$$\lim_{L_t \to 0} F_L(K_t, L_t, A_t) = \infty \quad \text{and} \quad \lim_{L_t \to \infty} F_L(K_t, L_t, A_t) = 0, \quad \forall K_t > 0.$$

These conditions are important for ensuring the existence of interior equilibria. All factors of production are necessary, i.e.,

$$F(0, L_t, A_t) = F(K_t, 0, A_t) = 0.$$

Output is unbounded above, i.e.,

$$\lim_{K_t\to\infty} F(K_t, L_t, A_t) = \infty \quad \text{and} \quad \lim_{L_t\to\infty} F(K_t, L_t, A_t) = \infty.$$

3.3 Production Function in Intensive Form

Let $y_t \equiv Y_t/L_t$ and $k_t \equiv K_t/L_t$ denote output and capital per worker, respectively. The production function F is homogeneous of degree 1, thus:

$$y_t = \frac{F(K_t, L_t, A_t)}{L_t} = F\left(\frac{K_t}{L_t}, 1, A_t\right) \equiv f(k_t).$$

By the definition of f and properties of F, we have the following:

- $f(0) = 0$ and f is twice differentiable.
- $\lim_{K_t\to 0} f'(k_t) = \infty \quad \text{and} \quad \lim_{K_t\to\infty} f'(k_t) = 0.$
- $F_K(K_t, L_t, A_t) = f'(k_t)$ and $F_L(K_t, L_t, A_t) = f(k_t) - kf'(k_t)$.
- f is increasing, $f'(k_t) > 0$ and concave, $f''(k_t) < 0$.

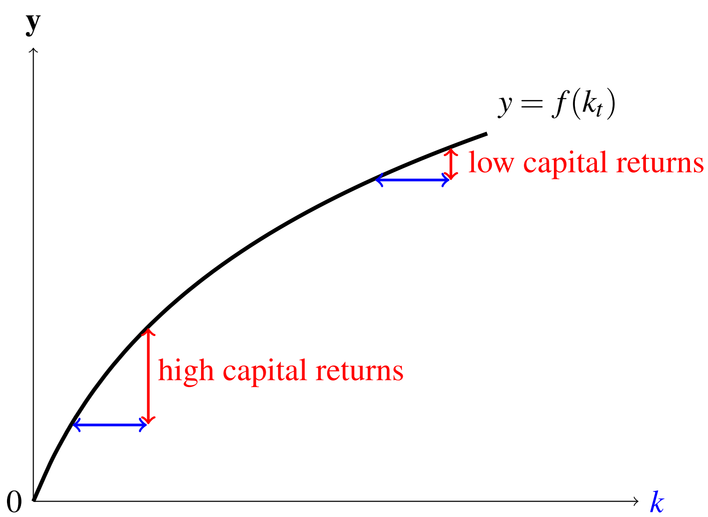

Figure 1: Production function: diminishing marginal returns

4 Market Structure, Endowments, and Equilibrium

In this section, we describe the market structure, endowments, and equilibrium conditions in the Solow model.

4.1 Competitive Markets

In the Solow model, markets are assumed to be competitive:

- Households and firms are price takers, meaning they accept market prices as given and cannot influence them.
- Prices equilibrate the markets, ensuring that supply equals demand.

4.2 Labor Market Equilibrium

At each date t, households have $\overline{L}_t$ units of labor that they supply inelastically. The labor market equilibrium condition at time t can be expressed as:

$$L_t = \overline{L}_t$$

where L_t is the labor demand by firms. Given the wage w_t, this equilibrium condition can also be written as:

$$L_t \leq \overline{L}_t, \quad w_t > 0 \quad \text{and} \quad (L_t - \overline{L}_t)w_t = 0.$$

4.3 Capital Market Equilibrium

Households also own capital and supply the quantity $\overline{K}_t > 0$, with $K(0) > 0$ being the initial capital stock. The equilibrium condition for the capital market is:

$$K_t = \overline{K}_t.$$

Capital depreciates at rate δ, meaning that of 1 unit of capital used at time t, only $1-\delta$ units remain for the next period. Let R_t be the real rent on capital at time t. The interest rate received by households will be:

$$r_t = R_t - \delta.$$

4.4 Firm's Profit Maximization

The representative firm maximizes its profit by choosing the quantity of production factors:

$$\max_{L_t \geq 0, K_t \geq 0} F(K_t, L_t, A_t) - w_t L_t - R_t K_t.$$

The first-order conditions imply:

$$R_t = F_K(K_t, L_t, A_t) = f'(k_t), \tag{2}$$

$$w_t = F_L(K_t, L_t, A_t) = f(k_t) - k_t f'(k_t). \tag{3}$$

At the equilibrium of the Solow model, firms make zero profit, and in particular:

$$Y_t = w_t L_t + R_t K_t. \tag{4}$$

This immediately follows from Euler's theorem for $m = 1$.

5 Variables Dynamics

5.1 Capital Stock Movement Law

The law of motion for the capital stock is:

$$K_{t+1} = (1-\delta)K_t + I_t,$$

where I_t is the investment at time t. The equation shows that the capital stock in the next period, K_{t+1}, is determined by the level of investment, I_t, which replaces or expands capital, and the remaining portion of the current capital, $(1-\delta)K_t$, after accounting for the depreciated quantity, δK_t.

Since the economy is closed, the sum of global consumption and investment cannot exceed total production:

$$Y_t = C_t + I_t.$$

Since the economy is closed (and there is no government spending):

$$S_t := Y_t - C_t = I_t.$$

5.2 Law of Motion of Saving

Households save a constant fraction s of their income. This can be expressed as:

$$S_t = sY_t.$$

Consequently, consumption C_t is given by:

$$C_t = Y_t - I(t) = (1-s)Y_t. \tag{5}$$

Thus, the **fundamental equation of the Solow model** is:

$$K_{t+1} = sY_t + (1-\delta)K_t = sF(K_t, L_t, A_t) + (1-\delta)K_t. \tag{6}$$

This is a nonlinear difference equation. The equilibrium of the Solow model is described by this equation, along with the laws of motion for L_t and A_t.

5.3 Law of Motion of Employment (Population)

The demographic growth rate, denoted n, is exogenous and constant. Thus, at each date t, the number of households is given by:

$$\overline{L}(t) = (1+n)\overline{L}(t-1) \quad \Longleftrightarrow \quad L_t = (1+n)^t \overline{L}(0),$$

with $\overline{L}(0)$ being the initial population level. We normalize $\overline{L}(0) = 1$.

6 Equilibrium and Steady State

6.1 Equilibrium

Definition 6.1 (Equilibrium)**.** *Given a sequence of $\{L_t, A_t\}_{t=0}^{\infty}$ and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, production levels, consumption levels, wages, and rental rates of capital $\{K_t, Y_t, C_t, w_t, R_t\}_{t=0}^{\infty}$ such that K_t satisfies the fundamental equation of the Solow model (6), Y_t is given by (1), C_t is given by (5), and w_t and R_t are given by (3) and (2), respectively.*

Equilibrium is defined as a complete trajectory of allocations and prices. It does not refer to a static object but rather specifies the trajectory of economic variables over time.

6.2 Steady State in the Solow Model

Initially, assume that the productivity $A(t) = A$ is constant and exogeneous. Recall the fundamental dynamic equation of Solow:

$$K_{t+1} = sF(K_t, L_t, A_t) + (1-\delta)K_t. \tag{7}$$

Dividing this equation by L_t, we obtain:

$$(n+1)k_{t+1} = sf(k_t) + (1-\delta)k_t. \tag{8}$$

This equation indicates that saving serves to replace depreciated capital, endow new workers with capital, and potentially increase the capital per worker. Equation (8) is also called the difference equation of the Solow model equilibrium. The other equilibrium quantities can be derived from k_t.

Definition 6.2 (Steady-State Equilibrium)**.** *A steady-state equilibrium (without technological progress), also called a regular state, is an equilibrium path in which:*

$$k_t = k^*, \textit{ for all } t.$$

The economy will tend toward this steady-state equilibrium over time (but it will never reach it in finite time). At the steady state, all aggregate variables (L_t, K_t, Y_t, C_t) and prices (w_t, R_t) grow at a constant rate, possibly zero.

The expressions for output and consumption per worker at the steady state are:

$$y^* = f(k^*) \quad \text{and} \quad c^* = (1-s)f(k^*).$$

Proposition 6.1. *The economy reaches a unique steady state such that :*

- k^* *exists and is unique, satisfying:*

$$\frac{f(k^*)}{k^*} = \frac{\delta + n}{s}.$$

- k^* *and* y^* *increase with* s *and decrease with* δ *and* n*.*
- c^* *is non-monotonic with respect to* s *and decreases with* δ *and* n*.*

Proof. k^* is the steady-state capital if and only if it solves:

$$(n+1)k^* = sf(k^*) + (1-\delta)k^*.$$

This implies that:

$$\phi(k^*) \equiv \frac{f(k^*)}{k^*} = \frac{\delta + n}{s}.$$

The properties of f imply that ϕ is continuous and strictly decreasing, with:

$$\phi'(k) = \frac{f'(k)k - f(k)}{k^2} = -\frac{F_L}{k^2} < 0,$$

$$\phi(0) = f'(0) = \infty \quad \text{and} \quad \phi(\infty) = f'(\infty) = 0.$$

Thus, the equation has a unique solution:

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right).$$

Recall that:

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right).$$

By applying the formula for the derivative of the inverse function:

$$(\phi^{-1}(k))' = \frac{1}{\phi'[\phi^{-1}(k)]},$$

we find that k^* decreases with $(\delta+n)/s$ (since $\phi'(k)<0$).

Since $y^* = f(k^*)$ and f is increasing, y^* also decreases with $(\delta+n)/s$.

On the other hand, consumption is given by:

$$c^* = (1-s)f(k^*).$$

Thus, c^* decreases with $\delta+n$, but the effect of s is ambiguous.

□

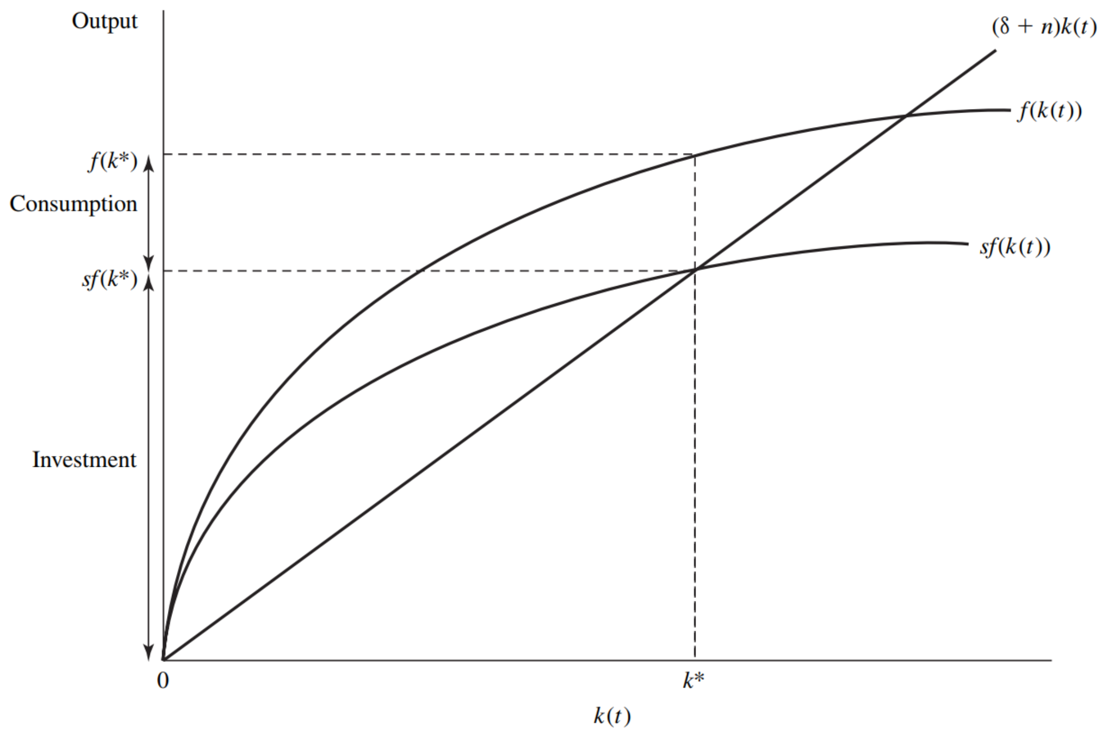

Figure 2: Investment and consumption in the Solow steady state

6.3 Golden Rule

For each value of the savings rate s, there exists a unique value of k^*, and $\partial k^*(s)/\partial s > 0$. The steady-state consumption can be written as:

$$c^*(s) = f(k^*(s)) - (n+\delta)k^*(s).$$

$c^*(s)$ is increasing for low values of s and decreasing for high values of s.

$$\frac{\partial c^*(s)}{\partial s} = \left[f'(k^*(s)) - (n+\delta)\right] \frac{\partial k^*(s)}{\partial s}.$$

Let s_{or} denote the savings rate that maximizes consumption, and k_{or} the corresponding capital level. Then:

$$f'(k_{or}) = (n+\delta).$$

6.4 Golden Rule and Dynamic Inefficiency

In the Solow model, the savings rate is exogenous. An important question is what the appropriate savings rate should be. It is difficult to answer this until we have specified a detailed objective function (Chapter 3). One answer is to choose the savings rate that maximizes consumption, i.e., s_{or}. A savings rate $s \geq s_{or}$ is inefficient because higher consumption per capita could be achieved by reducing the savings rate.

Suppose an economy with a saving rate $s_2 > s_{or}$, then $k_2^* > k_{or}$ and $c_2^* < c_{or}$. If we reduce s_2 to s_{or}, per capita consumption first increases and then decreases monotonically during the transition to its new steady-state value, c_{or}. Since $c_2^* < c_{or}$, consumption will be greater than c_2 at all moments during the transition and in the new steady-state. Therefore, when $s > s_{or}$, the economy is in a situation of over-saving in the sense that per capita consumption at all points in time could be increased by reducing the saving rate. An economy that over-saves is said to be dynamically inefficient because its per capita consumption trajectory is lower than potential alternative trajectories at any point in time.

If the economy has a saving rate $s_1 < s_{or}$, then higher savings will increase per capita consumption in the steady state. However, this increase in the saving rate will reduce consumption today and for some time during the transition period. The result will therefore be seen as good or bad depending on how households weigh today's consumption against future consumption trajectories. We cannot judge whether an increase in the saving rate is appropriate in this situation until we have made specific assumptions about how agents discount the future.

6.5 Dynamic Transition

The study of the transition dynamics answers the question: **Does an economy that is not at its steady state converge towards it?** In other words: **Will the economy eventually return to its steady state if an exogenous shock drives it away?**

Proposition 6.2. *For any strictly positive initial level of capital per worker k_0, the economy converges towards its steady-state level.*

- *The growth rate is positive and decreases to zero if $k_0 < k^*$.*
- *The growth rate is negative and increases to zero if $k_0 > k^*$.*

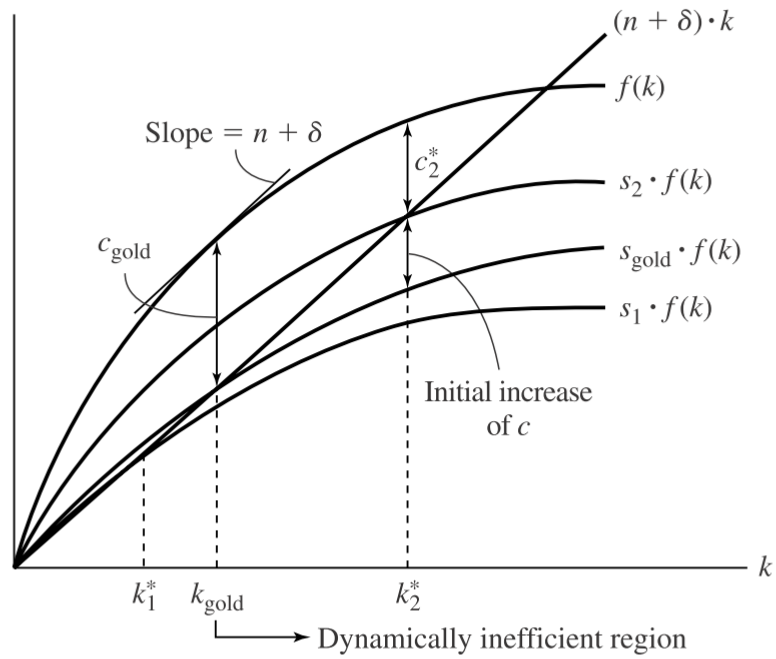

Figure 3: Golden Rule and Dynamic Inefficiency (Barro & Sala-i-Martin, p. 36)

Proof. Let h be the function:

$$h(k) = \frac{sf(k) + (1-\delta)k}{n+1}.$$

h is strictly increasing and concave (and continuous and differentiable):

$$h'(k) = \frac{sf'(k) + (1-\delta)}{n+1} > 0 \quad \text{and} \quad h''(k) = \frac{sf''(k)}{n+1} < 0.$$

We show that if $k_0 < k^*$ then k_t increases and stays to the left of k^*, i.e.,

$$k_t < k^* \text{ and } \gamma_t \equiv \frac{\Delta k_t}{k_t} > 0, \forall t.$$

Since h is increasing, it is enough to apply h t times to the inequality $k_0 < k^*$ and obtain:

$$k_t < k^*.$$

Moreover,

$$\gamma_t = \frac{k_{t+1} - k_t}{k_t} = \frac{s}{n+1}\left[\frac{f(k_t)}{k_t} - \frac{n+\delta}{s}\right].$$

Recall that ϕ is decreasing:

$$k_t < k^* \implies \phi(k_t) - \frac{n+\delta}{s} > \phi(k^*) - \frac{n+\delta}{s} = 0.$$

This shows that when $k_0 < k^*$, k_t is increasing.

Thus, when $k_0 < k^*$, k_t grows asymptotically towards k^*.

Since $k_{t+1} = h(k_t)$, and k_t converges, its limit is the unique fixed point of h, which is nothing but k^* by definition.

Symmetrically, it can be shown that if $k_0 > k^*$, then k_t decreases asymptotically towards k^*.

□

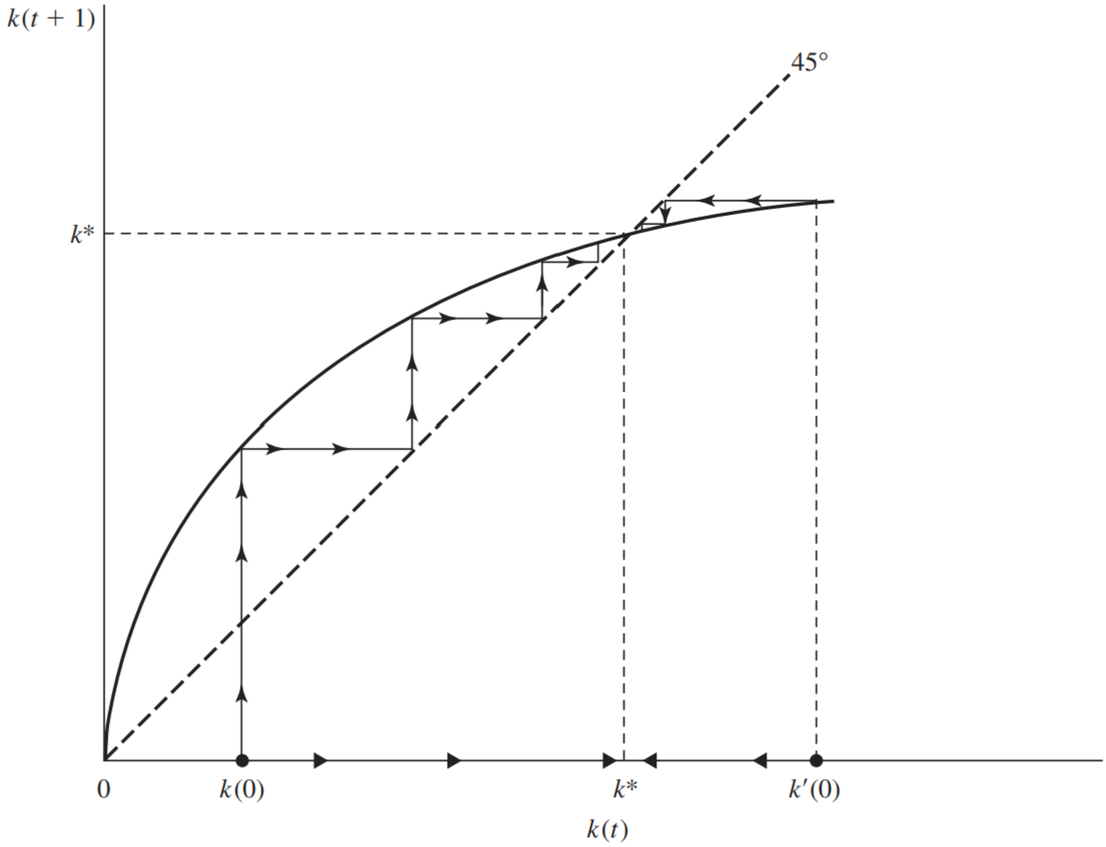

Figure 4: Transition Dynamics in the Solow Model

7 Solow Model in Continuous Time

Consider the difference equation:

$$x(t+1) - x(t) = g(x(t)). \tag{9}$$

where for each unit of time $t = 0, 1, 2, \ldots$, $g(x(t))$ describes the absolute growth of x between t and $t+1$.

Equation (9) describes the variation of the variable x between two discrete points in time, t and $t+1$. Now consider the following approximation for $\Delta t \in [0, 1]$:

$$x(t + \Delta t) - x(t) \simeq \Delta t g(x(t)).$$

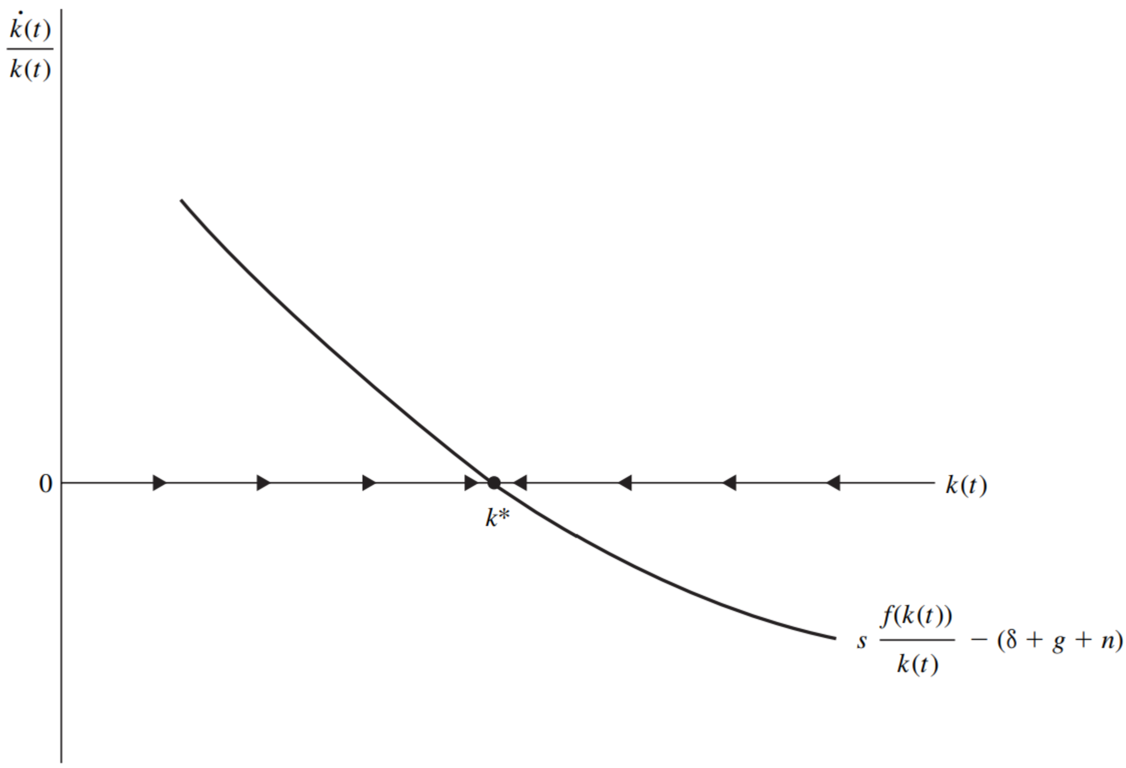

Figure 5: Capital-Labor Ratio in the Solow Model

If $\Delta t = 0$, equation (7) is just an identity, and if $\Delta t = 1$, we obtain equation (9).

This relation is a linear approximation between $\Delta t = 0$ and $\Delta t = 1$. This approximation will be relatively accurate if the distance between t and $t+1$ is not very large, so that $g(x) = g(x(t))$ for all $x \in [x(t), x(t+1)]$.

By dividing both sides of (7) by Δt and taking the limit, we get:

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \dot{x}(t) \approx g(x(t)). \tag{10}$$

where $\dot{x}(t) \equiv \dfrac{\mathrm{d}x(t)}{\mathrm{d}t}$.

Equation (10) is a differential equation representing the same dynamics as the difference equation (9).

7.1 Fundamental Equation of Solow: Continuous Time

In the continuous-time version of the Solow model, the production side remains unchanged. The equations (3) and (2) still give the factor prices, but now $w(t)$ and $R(t)$ are interpreted as the instantaneous wage rate and the rental rate of capital.

Savings is given by:

$$S(t) = sY(t).$$

Consumption is given by:

$$C(t) = (1-s)Y(t). \tag{11}$$

The labor factor (population) in continuous time is:

$$L(t) = L(0)\exp(nt). \tag{12}$$

7.2 Equilibrium: Solow Model in Continuous Time

Recall that the capital stock per worker is:

$$k(t) \equiv \frac{K(t)}{L(t)}.$$

Then:

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)} = \frac{\dot{K}(t)}{K(t)} - n.$$

The fundamental equation of the Solow model in continuous time becomes:

$$\dot{K}(t) = sF(K(t), L(t), A) - \delta K(t).$$

Using the properties of F, we get:

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + n). \tag{13}$$

Definition 7.1. *Given the initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, labor, production levels, consumption levels, wages, and capital rental rates $\{K(t), L(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$ such that $L(t)$ satisfies (12), $k(t) \equiv K(t)/L(t)$ satisfies (13), $Y(t)$ is given by the aggregate production function, $C(t)$ is given by (11), and $w(t)$ and $R(t)$ are given by (3) and (2).*

As before, the steady-state equilibrium implies that $k(t)$ remains constant at a certain level k^*:

$$k^* = \phi^{-1}\left(\frac{\delta + n}{s}\right) \quad \text{with} \quad \phi(k) = \frac{f(k)}{k}.$$

7.3 Dynamic Transition: Continuous Time

Proposition 7.1. *For an initial capital level $k(0) > 0$, the economy described by the Solow model in continuous time without technological change converges asymptotically to its steady state.*

- $\dot{k}(t)/k(t) > 0$ *and decreases toward zero if* $k(0) < k^*$.

- $\dot{k}(t)/k(t) < 0$ *and increases toward zero if* $k(0) > k^*$.

The proof is identical to that provided for the discrete time model.

The growth rate of output is:

$$\frac{\dot{y}}{y} = \frac{f'(k)}{f(k)}k = sf'(k) - (\delta + n)\,\alpha_k \tag{14}$$

where $\alpha_k(t) \equiv f'(k(t))k(t)/f(k(t))$ is the share of capital income.

By differentiating equation (14) with respect to k, we find:

$$\partial(\dot{y}/y)/\partial k = \left[\frac{f''(k)\cdot k}{f(k)}\right](\dot{k}/k) - \frac{(n+\delta)f'(k)}{f(k)}(1-\alpha_k)\,.$$

- If $k(0) < k^*$, then $\dot{k}/k > 0$ and thus $\partial(\dot{y}/y)/\partial k \geq 0$.
- If $k(0) > k^*$, the sign of $\partial(\dot{y}/y)/\partial k$ is ambiguous, but it will be negative as we approach the steady state.

8 The Solow Model with Technological Progress

The production function (Uzawa's Theorems) is:

$$Y(t) = F(K(t), A(t)L(t)),$$

where technological progress makes labor more productive. Technological progress evolves at the rate $g > 0$, such that:

$$g = \frac{\dot{A}(t)}{A(t)}.$$

The population continues to grow at the rate $n > 0$, with $n = \dot{L}(t)/L(t)$. Therefore, the fundamental equation of the Solow model becomes:

$$\dot{K}(t) = sF(K(t), A(t)L(t)) - \delta K(t). \tag{15}$$

Let us now define $k(t)$ as capital per effective unit of labor, that is:

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \tag{16}$$

Differentiating this expression with respect to time:

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n. \tag{17}$$

The output per effective unit of labor can be written as follows:

$$\hat{y}(t) \equiv \frac{Y(t)}{A(t)L(t)} = F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \equiv f(k(t)).$$

The income per capita is then:

$$y(t) \equiv Y(t)/L(t) = A(t)\hat{y}(t) = A(t)f(k(t)).$$

Thus, if $\hat{y}(t)$ is constant, income per capita, $y(t)$, will increase over time because $A(t)$ is increasing. We can no longer talk about a steady state where income per capita is constant. We now seek a balanced growth path, where income per capita increases at a constant rate. Some transformed variables, such as $\hat{y}(t)$ or $k(t)$ in (16), remain constant along the balanced growth path.

The terms "steady state", "regular state", and "balanced growth path" are used interchangeably. The equations (15) and (17) combined imply:

$$\frac{\dot{k}(t)}{k(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g + n).$$

Which can also be written as:

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n).$$

The only difference is the presence of g. Thus, k is no longer capital per labor, but capital per effective labor.

8.1 Equilibrium: Solow Model with Technological Progress

Proposition 8.1. *A steady-state equilibrium (with technological progress and population growth) is a balanced path where:*

$$k(t) = k^\star,$$

where $k^\star$ satisfies the following relation for all t:

$$\frac{f(k^\star)}{k^\star} = \frac{\delta + n + g}{s}.$$

Production and consumption per worker grow at the rate g.

8.2 Dynamic Transition

Proposition 8.2. *For an initial level of capital per effective unit of labor $k(0) > 0$, the economy described by the Solow model with technological progress and population growth will* *asymptotically converge to its steady state:*

$$k(t) \to k^{\star}.$$